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ABSTRACT

The third volume of the 24th annual conference of the International Group for the Psychology of Mathematics Education contains full research report papers. Papers include: (1) "Mathematics classrooms functioning as communities of inquiry: Possibilities and constraints for changing practice" (Susie Groves, Brian Doig, and Laurance Splitter); (2) "Thinking of new devices to make viable symbolic calculators in the classroom" (Dominique Guin and Luc Trouche); (3) "Some theoretical problems of the development of mathematical thinking" (Valery A. Gusev and Ilder S. Safuanov); (4) "The role of figures in geometrical proof-problem solving: Students' cognitions of geometrical figures in France and Japan" (Kouhei Harada, Elisabeth Gallou-Dumiel, and Nobuhiko Nohda); (5) "Thinking about the discursive practices of teachers and children in a 'National Numeracy Strategy' lesson" (Tansy Hardy); (6) "Classroom discussion on the representation of quantity by fractions: Stability of misconception and implications to practice" (Junichi Hasegawa); (7) "Process of internalizing new use of multiplication through classroom instruction: A case study" (Keiko Hino); (8) "Understanding linear algebraic equations via super-calculator representations" (Ye Yoon Hong, Mike Thomas, and Oh-Nam Kwon); (9) "The relation of Mozambican secondary school teachers to a mathematical concept: The case of limits of functions" (Danielle Huillet and Balbina Mutemba); (10) "The meaning of terms concerning the time ordering for first grade students: The influence of cultural background" (Sonia Igliori, Cristina Maranhao, and S. Sentelhas); (11) "The relationships between fluency and flexibility of divergent thinking in open ended mathematics situation and overcoming fixation in mathematics on Japanese junior high school students" (Toshihiro Imai); (12) "A case study of student emotional change using changing heart rate in problem posing and solving Japanese classroom in mathematics" (Masami Isoda and Akemi Nakagoshi); (13) "Using students' statistical thinking to inform instruction" (Graham A. Jones, Cynthia W. Langrall, Carol A. Thornton, Edward S. Mooney, Arsalan Wares, Bob Perry, Ian J. Putt, and Steven Nisbet); (14) "The student experience of online mathematics enrichment" (Keith Jones and Helen Simons); (15) "Epistemological features in the mathematics classroom: Algebra and geometry" (Maria Kaldrimidou, Haralambos Sakonidis,

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and Marianna Tzekaki); (16) "What do we remember when it's over? Adults' recollections of their mathematical experience" (Ronnie Karsenty and Shlomo Vinner); (17) "Acquiring the concept of derivative: Teaching and learning with multiple representations and CAS" (Margaret Kendal and Kaye Stacey); (18) "Year 6 students' progression on the core mathematics course of an integrated learning system" (Gillian C. Kidman, Annette R. Baturo, Tom J. Cooper, and Romina Proctor); (19) "Solving multiplicative word problems with decimal fractions: The effects of proportional reasoning and metacognition" (Tadayuki Kishimoto); (20) "Does a research based teacher development program affect teachers' lesson plans?" (Ronith Klein and Dina Tirosh); (21) "A research on the validity and effectiveness of 'two-axes process model' of understanding mathematics at elementary school level" (Masataka Koyama); (22) "The effects of different instructional methods on the ability to communicate mathematical reasoning" (Bracha Kramarski); (23) "Narrative elements in mathematical argumentations in primary education" (Gotz Krummheuer); (24) "Changing teaching methods in school mathematics: An analysis of some episodes from classes" (Marie Kubinova, Jiri Mares, and Jarmila Novotna); (25) "Development of concepts for division in third grade teaching experiments: From the viewpoint of the dual nature of concepts and symbolizing processes" (Koichi Kumagai); (26) "Moving between mixed-ability and same-ability settings: Impact on learners" (Bilha Kutscher and Liora Linchevski); (27) "A developmental assessment of pupils use of domain-specific and general strategies in problem solving" (Leonidas Kyriakides); (28) "Coding the nature of thinking displayed in responses on nets of solids" (Christine Lawrie, John Pegg, and Angel Gutierrez); (29) "A comparison between Malaysian and United Kingdom teachers' and students' images of mathematics" (Chap Sam Lim); (30) "On developing teachers knowledge by using cases constructed by researcher and classroom teachers" (Pi-Jen Lin); (31) "Indispensable mathematical knowledge-IMK and differential mathematical knowledge-DMK: Two sides of the equity coin" (Liora Linchevski, Bilha Kutscher, Alwyn Olivier, and Kate Bennie); (32) "Problem posing as a tool for learning, planning and assessment in the primary school" (Tom Lowrie and Jacquelyn Whitland); (33) "Length measurement: Conventional units articulated with arbitrary ones" (Cristina Maranhao and T. Campos); (34) "Dynamic transformations of solids in the mathematics classroom" (Christos Markopoulos and Despina Potari); (35) "States of understanding relations among concepts of geometric figures: Considered from the aspect of concept image and concept definition" (Nanae Matsuo); (36) "Supporting preservice teachers' understanding of place value and multidigit addition and subtraction" (Kay McClain and Janet Bowers); (37) "Spatial abilities in primary schools" (Hartwig Meissner and Guido Pinkernell); (38) "Analysis and synthesis of the Cartesian product by kindergarten children" (Ibby Mekhmandarov); and (39) "The 'conflicting' concepts of continuity and limit: A conceptual change perspective" (Kaarina Merenluoto and Erno Lehtinen). (ASK)

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Proceedings

of the

24th Conference

of the

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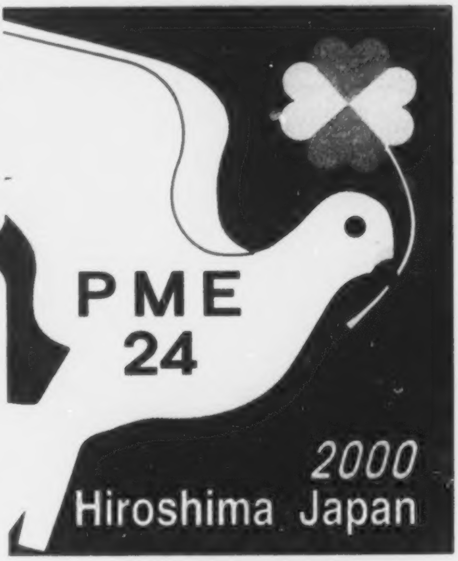

July 23-27, 2000

Editors: Tadao Nakahara
and Masataka Koyama
Hiroshima University

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RESEARCH REPORTS

MATHEMATICS CLASSROOMS FUNCTIONING AS COMMUNITIES OF INQUIRY:

POSSIBILITIES AND CONSTRAINTS FOR CHANGING PRACTICE

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As part of a study of current models of Australian primary mathematics practice and the extent to which these support mathematics classrooms functioning as communities of inquiry, three groups of educators viewed videotape of Australian and Japanese lessons, engaged in lengthy discussion and provided written feedback. There was a high level of agreement among principals, teachers and mathematics educators with the notion of mathematics classrooms functioning as communities of inquiry, together with a realisation that current Australian practice falls far short of this goal. They also recognised many aspects of communities of mathematical inquiry in videotape from the Japanese primary classroom.

Introduction

> Improvement [in teaching] will not happen by itself. It will require designing and building a research-and-development system that explicitly targets steady, gradual improvement of teaching and learning.
>
> (Stigler & Hiebert, 1999, p. 131)

The work reported here[1] is part of an ongoing program of research and development into models of primary mathematics practice consistent with classrooms functioning as communities of inquiry (see, for example, Doig, 1997; Groves, 1997; Groves & Doig, 1998). Our work is based on the notion of communities of inquiry, which underpins the *Philosophy for Children* movement (see, for example, Splitter & Sharp, 1995; Splitter, in press). Key features of classrooms functioning as communities of *philosophical* inquiry are the development of skills and dispositions associated with good thinking, reasoning and dialogue; the use of subject matter which is conceptually complex and intriguing, but accessible; and a classroom environment characterised by a sense of common purpose, mutual trust and risk-taking. Our concern is how these features can be made a part of everyday classroom practice in primary mathematics.

---

[1] The work reported here was carried out as part of the Australian Research Council funded project *Mathematics classrooms functioning as communities of inquiry: Models of primary practice.*

While there have been a number of research and development projects which can be characterised as reflecting communities of inquiry in mathematics (see, for example, Lampert, 1990; Cobb, Wood & Yackel, 1991; Ball, 1993; Brown & Renshaw, 1995; Gravemeijer, McClain & Stephan, 1998), when teachers in the wider education community attempt to implement these ideas in their own classrooms, they often adopt only the superficial features (see, for example, Knuth, 1997; Stigler & Hiebert, 1999). Commenting on the Third International Mathematics and Science Study (TIMSS) video study of 100 German, 81 United States and 50 Japanese year 8 classrooms, Stigler et al (1999) state that, while there is a high level of agreement with reform ideas among US teachers, there is little evidence that these are actually operationalised. In contrast, Japanese lessons "include high-level mathematics, a clear focus on thinking and problem solving, and an emphasis on students deriving alternative solution methods and explaining their solutions" (p. vii).

A key issue therefore is: How can important findings of research be translated into a model of classroom practice which is accessible to the wider teaching community? What could a new, widely applicable, model of mathematics teaching, which adopts the aim of fostering students' constructions of powerful mathematical ideas through a process of confronting problematic situations, reflective thinking and reasoning, and participating in whole class dialogue and argumentation, look like?

Stigler and Hiebert (1999) argue that because teaching is a cultural activity, change needs to be continual, gradual and incremental. According to Yackel (1994), a first priority for changing teaching practice is to make problematic for teachers aspects of their current practice. Stigler (American Federation of Teachers & National Centre for Educational Statistics, 1998) claims that it is because discussions of teaching take place outside of the context of actual examples that "caricatures of different styles of teaching that don't really exist … [lead to] emotional debates over how you should teach". He goes on to say: "Let's look at examples and let's say exactly what it is about this that you'd like to see changed. That's how we come to understand what good teaching is. We haven't had this conversation in this country [the USA]".

The *Mathematics classrooms functioning as communities of inquiry: Models of primary practice* project attempted to have such a conversation in Australia by examining current models of Australian mathematics practice and investigating the extent to which these support or hinder mathematics classrooms functioning as communities of inquiry. It further sought to determine the extent to which the wider local education community (primary teachers and principals, as well as mathematics teacher educators and consultants) endorse the goal of mathematics classrooms functioning as communities of inquiry which has been explicated by so many mathematics educators world-wide. Videotape from a local Japanese school was used as a counterpoint to that obtained from local Australian classrooms.

Methodology

In order to establish what constitutes current primary mathematics practice in Australia, video and other data were collected from a stratified random sample of ten year 3 and 4 classrooms in the state of Victoria. One mathematics lesson of approximately one hour's duration was videotaped in each of the ten classrooms and an outline of the aims for each lesson, as well as copies of any work-sheets used by the children, were collected. Similar data were collected from the year 3 and the year 4 classes at the Japanese School of Melbourne. This school, which is operated by the Japanese Government along identical lines to schools in Japan, offers tuition to students who are short-term residents of Australia and who can expect to return to their normal Japanese classrooms at any time during the year with no disruption to their education. Based on previous lessons observed at this school (Groves, 1998) we would confirm that the model of classroom practice conforms with that described by Ito-Hino (1995), Stigler et al (1999) and observed in 1993 by Groves and Doig in Tsukuba, Japan.

An analysis of the videotapes was carried out, using a framework based on that developed by Schmidt et al (1996), who use the term "characteristic pedagogical flow" to describe recurrent patterns of observable characteristics in a set of lessons. Based on our observations, fieldnotes and this analysis, three edited tapes of up to 10 minutes each were produced, representing the contrasting characteristic pedagogical flows captured on the Australian video tapes. A similar 8-minute (sub-titled) videotape was produced for the Japanese grade 3 lesson.

These short videotapes were used as a stimulus for three separate four-hour focus group meetings for randomly selected teachers (n=12), principals (n=6) and mathematics teacher educators and consultants (n=10). Discussions were based on the findings from the analysis of the ten Australian and two Japanese lessons and a viewing of the four edited videotapes.

The first two hours of each meeting addressed the extent to which the participants believed that the Australian videotapes reflected dominant models of current Australian practice. Participants were provided with the framework used in the analysis and were asked to focus on the major structural features identified.

The second half of each meeting began with a brief explanation of the notion of communities of *philosophical* inquiry and how they might apply in mathematics. Particular emphasis was placed on the three key features outlined in the introduction to this paper — namely good thinking and dialogue; the use of conceptually complex and intriguing problematic situations; and an appropriate classroom environment. A brief discussion, intended to elicit participants' support for the notion of mathematics classrooms functioning in this way, followed. The Japanese videotape was then shown, after which a discussion was held to determine the extent to which participants perceived characteristic features of Australian and Japanese practice as supporting or hindering mathematics classrooms functioning as communities of inquiry.

The researchers took extensive notes of the discussions, which were also tape recorded for later transcription. In addition, the last fifteen minutes of each "half" of the meeting were devoted to participants completing written responses to a list of "prompts" in order to provide data on individual views.

Results from the analysis of data relating to what constitutes the dominant models of current Australian practice will be presented elsewhere. This paper focuses on the extent of participants' support for the notion of mathematics classrooms functioning as communities of inquiry and characteristic features of Australian and Japanese practice which were seen as supporting or hindering mathematics classrooms functioning in this way. The major data source for this paper is the participants' comments, written at the conclusion of the second half of the meeting. (At this time, one teacher and one mathematics educator had already left, so the number of respondents in the analysis below does not correspond to the numbers given earlier).

Three prompts were provided to facilitate these written comments, although participants were free to ignore or supplement these. The prompts were:

- *Is it desirable to encourage communities of mathematical inquiry in primary classrooms? ... because?*
- *Is it possible for mathematics classrooms to function as communities of inquiry?*
- *How could this be achieved? What would need to change? What should be kept as is?*

The written comments confirmed the points made in the discussions, however they did not necessarily address all aspects of the discussion and hence reference will also be made here to participants' verbal comments.

Results

Results from the analysis of the written responses are presented here in the form of descriptive comments and selected representative quotes from participants.

All eleven teachers said they thought it desirable for mathematics classrooms to function as communities of inquiry. Reasons given included that communities of mathematical inquiry would lead to increased conceptual understanding, improve children's thinking skills, "allow success for children of different abilities", and lead to the "participation of children regardless of their ability". The group of principals responded positively, giving similar reasons to those given by teachers. However the principals added that a move to this approach would connect real life and other subject areas to mathematics; and would move away from current "limiting models of classroom practice". Like the other participants, mathematics educators agreed with the desirability of adopting communities of inquiry in mathematics. Reasons given repeated those given by the other groups, but also suggested that "the knowledge constructed and connections made [by children] are better" in classrooms functioning in this way.

Although all participants were in agreement that it is possible for mathematics classrooms to function as communities of inquiry, most qualified this by suggesting requirements necessary to make such a change in teaching practice possible. For example, teachers suggested that it would "require principals and curriculum co-ordinators to manage change by providing professional development". While principals' suggestions included the provision of support for teachers, they also suggested the need for a whole school focus on communities of inquiry in mathematics. The mathematics educators were in agreement with other participants, but their reservations about the possibility of implementing communities of inquiry in mathematics focused on teachers' mathematical competence and confidence.

In order to achieve communities of inquiry in mathematics, teachers suggested that they would require professional development programs, peer support and examples of such teaching in practice. However, the most common suggestion was for changes to the curriculum so that learning would be "process driven not based on outcomes only" and that a "less crowded curriculum" be adopted. The principals suggested that teachers need to see a reason to change their teaching practice, and that linking to "many of the good things that are currently happening" could provide a vehicle for such a change. The principals, like the teachers, expressed the need for a curriculum that was "not small step outcomes based". The mathematics educators made many suggestions how communities of inquiry in mathematics classrooms could be achieved. These included, not unexpectedly, increasing teacher content knowledge, professional development programs for teachers and the development of "a rich set of mathematical contexts". According to the mathematics educators, among the changes needed to current mathematics teaching practice were a non-outcomes-based curriculum; raising "the expectations in the [mathematics] teaching community"; and a change in "teachers' perceptions of mathematics".

Discussion

One of the major contrasts between the Australian and the Japanese lessons shown was the highly focused and conceptually oriented nature of the problematic situation presented to the children in the Japanese lesson. This led to considerable discussion and comment in all three focus groups. Teachers and principles contrasted the multitude of "tasks … not problems" in the Australian lessons — the "huge range of things explored" — with the genuinely problematic situation presented in the Japanese lesson and the fact that it "concentrated on one area in depth". The current outcomes based curriculum was believed to be a major obstacle to more conceptually focused teaching, with one mathematics educator writing that we "need to wean ourselves from dependence on ticking outcomes boxes" and one principal commenting during discussion that the "testing program dictates the curriculum". It was probably not surprising then that the most common theme, mentioned by participants in response to all three written prompts, was the need for changes to the primary mathematics curriculum.

The need to focus more on conceptual understanding was also evident from the written comments regarding the desirability of communities of mathematical inquiry in primary classrooms. These comments contained two main themes, the first being that communities of mathematical inquiry were desirable because they would "foster deeper, more complex understanding of concepts". The second theme stressed the importance of mathematical thinking — as one mathematics educator wrote, we want to "develop mathematical thinkers who are able to make a contribution to society".

Participants also recognised other aspects of communities of inquiry in the Japanese video — for example that the teacher was "in charge but not the authority", asking questions rather than providing information, while the children were prepared to take risks and were the ones coming to conclusions.

Written responses to the second prompt regarding the possibility of mathematics classrooms functioning as communities of inquiry suggest a difference of opinion between teachers and principals, on the one hand, and mathematics educators on the other. While all participants agreed that such change was possible, teachers and principals thought that professional development programs for teachers and a whole-school focus on communities of inquiry in mathematics were the major requirements for achieving this. Mathematics educators, on the other hand, focused on the need for teachers to "develop confidence and competence in their ability" and to "realise the real purpose of the content and structure of what they are teaching".

The third prompt, *How could this be achieved? What would need to change? What should be kept as is?* provided a wide range of comments. Behind the common theme of the constraints posed by the outcomes driven curriculum were sets of comments linked to specific groups of participants. Teachers' comments focused on the need for examples of communities of inquiry in mathematical in practice — "I think that teachers need to be convinced that *it works* " — on the need for professional development, and the need for "systemic change in the way staff view teaching and learning". Principals agreed with these views, and added that building on current successful practice, such as that in social education, would be of value. Teacher educators repeated suggestions for improved teacher content knowledge and for professional development programs. However, an additional suggestion was made to develop conceptually focused problematic situations to facilitate mathematical inquiry.

Conclusion

It is clear from the focus group discussions and the subsequent written comments from participants that there is overwhelming support among principals, teachers and mathematics educators for mathematics classrooms functioning as communities of inquiry, together with a realisation that current Australian practice falls far short of this goal. Participants also recognised many aspects of communities of mathematical inquiry in videotape from the Japanese primary classroom. Of particular note was the lack of conceptual focus in the Australian lessons. The fact that these views were almost unanimous among participants is particularly important, as the focus group members were selected randomly. This suggests that efforts to create opportunities for mathematics classrooms to function as communities of inquiry would not be in vain.

The major constraint on the development of more conceptually focused mathematics teaching was seen as the fragmented, outcomes based curriculum. We would agree with the notion that a fragmented curriculum prevents coherent, conceptually based teaching practices, and note the call for exemplars of conceptually focused problematic situations. Research into the key constituent features of problematic situations which can initiate and sustain dialogue needs to be undertaken, together with trialing in classrooms of examples of these by generalist primary teachers.

A disturbing, though minor, thread running through some of the participants' comments was the suggestion that *communities of inquiry* were already in existence in the guise of inquiry-based lessons. A belief that "inquiry" is the sole aspect of communities of inquiry removes the critical aspect of *community*, addresses neither the dialogical aspect nor the aspect of an environment of mutual trust and risk-taking. We believe that this may suggest why only superficial features of reforms are adopted.

Heartened by the results reported above, we intend to work with a small group of teachers, as a community of inquiry, to explore the extent to which generalist primary teachers can transform their mathematics classrooms into communities of inquiry. While this does not take up the challenge of systemic change to teaching practice, it will provide evidence of the possibility of mathematics classrooms functioning as communities of inquiry in a broad range of classrooms. This, we believe, is a necessary first step towards wider acceptance and implementation.

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THINKING OF NEW DEVICES TO MAKE VIABLE SYMBOLIC CALCULATORS IN THE CLASSROOM

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The introduction of symbolic calculators in the classroom influences on one hand the conceptualisation process, on the other hand the conception and the control of teaching situations. The teacher has to better understand the effects of instruments on the cognitive activity in order to master this variable instrument. Tranforming a calculator into a mathematical instrument both efficient for students and viable in the classroom requires to rethink the organisation of the space and the time of study. In this environment, the teacher's role is crucial in pointing out efficient techniques and to foster the acquisition of skills throughout the instrumental genesis.

I. INTRODUCTION

Since 1980, the use of graphic calculators has become an explicit aim in the French secondary mathematical curriculum. Nevertheless, no more than 15% of the teachers include these calculators in their teaching, but all students have ubiquitous use of them, because all types of calculators are freely used in secondary examinations; therefore students must acquire new techniques by themselves. Teachers'behaviour concerning these instruments may be analyzed as a sign of their non-viability in a given institution; in that case we must try to seek reasons of this situation.

In this educational context, Trouche and Guin (1996) have pointed out how the philosophy of 'seeing is reality' may influence the conceptualisation processes: screen images do not necessarily have a beneficial effect. These phenomena do not disappear with symbolic calculators; the comparison of students'behaviour in the two environments (Guin and Trouche 1999, p. 220) has revealed a larger dispersion of behaviour due to the bigger complexity of symbolic calculators. Furthermore, weaker students seem to give up any idea of understanding, while losing all control over results. Therefore, teachers should imagine new organisations in the classroom integrating symbolic calculators and design new situations in order to assist students during the instrumental genesis and more generally in their mathematical activity modified in this new environment.

II. THEORETICICAL FRAME AND METHODOLOGY

II.1 Theoretical frame

Research relies on different theoretical frames. On one hand, the cognitive ergonomy approach of the instrumented analysis is based on Vygotsky's hypothesis that cognitive capacities can be extended through interaction with the environment. Accommodating artificial systems may have an effect on cognitive development, knowledge construction and the nature itself of the knowledge generated (Verillon and Rabardel, 1995). The difference between the artefact (material object) and the instrument is pointed out: « the artefact becomes an instrument when the subject has been able to appropriate it for himself and has integrated him with his activity ». Since the instrument is not given, but must be worked out by the subject, he has to develop the instrumental genesis and efficient procedures in order to manipulate the artefact.

Artigue (1997) puts emphasis on the fact that material tools constrain the possibilities of interaction with mathematical objects and in this manner «they deeply condition the mathematics which can be produced and learnt». Therefore, technical constraints imposed by operating systems which are elements of the computational transposition (Balacheff 1993, p. 147) have to be analysed, because «Learning is based as much on these constraints as on the possibilities of investigations» (Dreyfus, 1993, p. 128). They imply, more or less explicitly, a prestructuration of the user's action (Guin and Trouche,1999, p.203) which induces specific procedures in students' adaptive processes.

Nevertheless, if calculators generate mathematical knowledge, students do not automatically receive much benefit from it: faced with unavoidable difficulties arising using these tools, some students select trial and error strategies without any reflective work relying upon their own mathematical knowledge (Guin et Delgoulet, 1996). Furthermore, even if students are able to produce powerful reflections within the computer environment, Noss and Hoyles (1996) have underlined the student's difficulties in these new ways of constructing meaning in building connections with the official mathematics outside of the technological environment.

On the other hand, this research relies on the Chevallard's approach of didactic phenomena (1992) which points out the impact of institutional values and norms relative to a mathematical object on the relationship the individual develops with respect to this object. Therefore, mathematical learning cannot be analysed only at a cognitive level, while disregarding the institutional context. Furthermore, Chevallard underlines the fact that the mediation of instruments in students'activity requires other mediations of the teacher in the classroom. He stresses the influence of official techniques, transmitted within teaching practices, on the personal knowledge construction.

This point of view leads to express questions in ecological terms: within one given educational system, is there a legitimacy of new technologies, how can they live, which organisation and which teacher practices related to these technologies are viable?

Calculators are personal tools, which may lead to an individualisation of the mathematical work in the classroom. In this context, the teacher has the responsibility to promote socialisation processes with the help of new devices integrating new instruments. Therefore, he (or she) has the crucial role in setting up the appropriate connections between the environment and official mathematical knowledge. In this way, psychological and social aspects are coordinated in the instrumental genesis and more generally throughout the learning process in which students reorganise their thinking from various interactions in the classroom.

II.2 Methodology

Relying on these theoretical frames, our methodology aims at investigating which organisation of learning activities can promote some cognitive reorganisation in our educational system. Situations are designed in order to take advantage of the constraints and discrepancies caused by calculators (Guin and Trouche 1999, p.202) and to enhance the coordination between the graphic and algebraic registers which constitutes a crucial point in the cognitive process (Duval 1996). The symbolic calculator deeply modifies ways of interacting with these registers, then situations are aiming to foster an experimental work based on these interactions and to encourage the research of a consistency between the various data obtained. Nevertheless, because such behaviour is not immediate among students in a calculator environment (see §I), symbolic calculators require thinking of new devices to promote it.

Regarding previous questions asked in ecological terms (see §II.1), the validation will therefore be internal. After an a priori analysis of modifications which are expected in students'work within the new environment (both including new situations and new organisations in the classroom), the validation derives from a comparison between the expected behaviour and the students'behaviour observed in the classroom. This comparison is analysed in terms of schemes which organise the subject's behaviour, because the conceptualisation process is based on these schemes (Vergnaud, 1990). Data collected for this research include the researchers' observation notes, individual interviews of selected students, questionnaires focusing on mathematical tasks and the students' relationship to the calculator, finally saved files and students written reports of working pairs.

III. DESIGNING A NEW STUDY ENVIRONMENT

The experiment reported herein was supported by the French Ministry of Education. The following describes the device experimented in two classes (10 Th and 12th grade or 15-16 and 17-18 year olds). Each student had the use of a TI-92 both at school and at home. Our device entailed a rearrangement of space and study times in various phases:

Throughout the theoretical lesson, the combination of the blackboard and the screen of one calculator enabled the individual student's work to be guided. Each student took a turn operating the projected calculator. This "sherpa-student" (see figure 1) played a central role as mediator: new relationships were established between the sherpa-student, the teacher and the other students.

Such organisation requires material conditions not available in all calculator experimental classrooms: for example, Doerr and Zangor (1999) pointed out the fact that their view screen did not allow for individual student's work to be projected. Our device enabled the teacher to become aware of the evolution of students instrumentation process. Moreover, the possibility to combine the screen and the blackboard favoured a specific work on the coordination between registers. This algebraic-graph-table switching entailed mathematical debates in the classroom, raising questions about calculator-generated results displayed by the teacher or the sherpa-student.

The practical sessions were inserted as one-hour weekly sessions. The students worked in pairs with their notebook and their calculator. Problem situations were created aimed at promoting interactions between calculators, theoretical results and hand-written calculations. After reflecting on these problems, each group had to explain and justify their comments, noting discoveries and dead-ends in a written research report. Throughout this phase, the teacher was a consultant giving hints and dealing with problems as they arose.

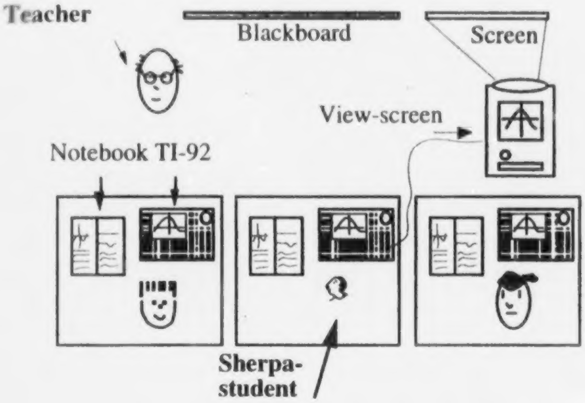

Figure 1. Organisation of a lesson in a calculator environment.

During the next session, a collective synthesis was organised with a projected calculator manipulated by a sherpa-student. After a comparison of various approaches of pairs, the teacher institutionalised and decontextualised the mathematical knowledge he (or she) wanted to retain among students productions. The synthesis phase was based on data collected during the practical sessions (written research reports and calculator working files).

IV. SOME RESULTS

The aim of this paper is not to report this whole experimentation, but to emphasise the role of the device in the viability of symbolic calculators in a given educational system. The following gives an overview of the change in student behaviour during the experiment.

It is notable that the calculator does not automatically induce a more questioning and reflective mathematical attitude among all students. Weaker students often give up the idea of understanding the command's meaning and what it does. We observed avoidance strategies of various forms (for example, random trials and zapping to other commands). Moreover, manipulations difficulties revealed conceptual difficulties as the recognition of two equivalent algebraic expressions, although students were relieved of the technical tasks. Difficulties were also identified in the conversion of representations from one register to another (graphic/ algebraic).

However, even if all students had not really undertaken experimental activities, not natural at all in the French educational context, the majority of students became aware of the possibilities of visualisation and interplay between registers. The instrumental genesis was characterised by two distinct phases: the first instrumentation level (discovery phase of various commands, their effects and their organisation) revealed a great diversity of strategies and techniques with few references to mathematical knowledge. The second phase occurred with a progressive awareness of the effective constraints and potential uses of the calculator, it is characterised by a pruning attitude towards the previous strategies and techniques and the development of a reasonable scepticism about calculator results, as in (Doerr and Zangor 1999). Students progressively acquire a conscious attitude while seeking mathematical consistency among all information sources (not only the calculator results). These results are concordant with the various levels of instrumentation described in (Defouad 99).

The analysis of instrumentation schemes (Trouche 2000) allowed to identify five behaviours profiles (random, mechanical, resourceful, theoretical and rational work methods) which highlight how change in student behaviour throughout the experiment is related to the student profile (Guin and Trouche 99). There is a gap between two groups of students: the effects seem to be favourable for students with sufficient mathematical background (rational, resourceful and theoretical work methods). Conversely for the other students, there is a significant dependence with regard to the symbolic calculator, consequently they cannot do anything without their 'crutch' and have difficulties to reach beyond the first instrumentation phase. The threshold of this stage can be precisely observed because it is the moment when the algebraic register takes priority over the graphic one, which then merely becomes a register for conjectures and checking.

Finally, the various questionnaires have pointed out for most students a decreasing trust in the calculator results, a real change in their relations to mathematics and their own self-confidence. Most students appreciated the use of the overhead projector with its possibilities of visualisation, manipulations and verifications. In the last set of questionnaires, students stressed the advantage of working in small groups for problem solving methods and experimental research.

Nevertheless, students awareness of the usefulness of our device was not immediate: for example, the written research report was seen at the beginning as a pointless constraint until students discover its crucial role to initiate debates in the collective synthesis.

V. DISCUSSION

How could one imagine new ways of organising the mathematical work to make viable symbolic calculators in the classroom? Firstly, an organisation class is based on some material conditions: for example, the sherpa system only works if all calculators are similar and can be connected to the same view screen.

Then the teaching organisation plays an essential role in pointing out efficient techniques and articulating them with older practices in the paper/ pencil environment. The additional time devoted to this task may facilitate access to effective instrumentation (Lagrange 1999). Moreover, we would like to notice that the device previously set out requires a teacher particularly skilled in using the symbolic calculator and especially aware of calculator's potentialities and constraints.

Nevertheless, we want to stress the fact that symbolic calculators will not be viable without some specific institutional conditions. In our experiments, some students had said they have learnt unnecessary things, simultaneously teachers tend to resist devoting their time to focus on efficient techniques: one cannot avoid the problem of the institution's lack of recognition of new skills acquired in the new environment. From an institutional point of view, such devices require an evolution in the mathematical community (more experimental conceptions, recognition of tools), in the curricula organisation and in teacher training. Finally, devices will evolve simultaneously with tools: the more the environment is complex, the more the human organisation plays a preponderant role.

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SOME THEORETICAL PROBLEMS OF THE DEVELOPMENT OF MATHEMATICAL THINKING

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This paper is a theoretical essay on problems of the research of the development of mathematical thinking. In mathematical education not only learning mathematical concepts by pupils, but also the development of their thinking abilities, mastering the means of mental activities are important. For this purpose, the appropriate motivation, fostering the characteristic traits of mathematical thinking, recognising the patterns of understanding, the use of the genetic approach to the mathematicas teaching might be helpful.

1. Introduction.

In this paper we will indicate and discuss those problems of the developing of mathematical thinking that, in our opinion, are most important today. This is a theoretical essay: we do not report investigation results but rather raise questions for further discussion and research.

We would not like to restrict our article to the boundaries of any definite (narrow) framework. Our purpose is to widely survey different directions in which the study of the development of mathematical thinking could be conducted. Nevertheless, most of our considerations will follow the activity approach elaborated by Soviet psychology.

We will discuss the development of mathematical thinking in the wider context of the general development of a personality. Under the general development we mean mainly mental development. In the problem of the development of a pupil we give special attention to the development of her/his thinking, because it is the thinking that determines all other intellectual functions: imagination, flexibility of the mind, latitude and depth of the thought etc.

There are several important conceptions of thinking in general and of mathematical thinking in particular. Most popular are theories of J. Piaget in the West, of L. Vygotsky and of S. Rubinshtein in the former U. S. S. R.

Rubinshtein (1958, p. 72) wrote: "The correct understanding of the interpretation of thinking as process assumes, that thinking is understood as activity of the subject interacting with the exterior world. *The thinking is a process just because it is continuous interaction of the man with the object...* "

S. L. Rubinshtein indicates to the connection between the process of thinking and its result: "Any thinking process has the resulting expression as a concept, new knowledge. Thus the thinking process is divided into the acts, each of which has a resulting expression - a "product". The latter is included into the further stages of a process. But the study of a "product" (image, concept, any knowledge) should not be a special subject of a psychological research. Psychology should research the *processual* thinking" (Ibid.).

2. The forming of means of mental activities: the role of awareness.

Developing pupils' thinking, it is necessary to constantly remember, that thinking consists both of knowledge and of abilities to think, i.e. of operations with knowledge. Therefore, as noted L. V. Vinogradova (1989), "it is necessary to teach the pupils to apply, to use their knowledge, i.e. to use their mental activity. The forming of such activity is one of main problems of teaching. The mastering of methods of intellectual activity by pupils means that they can operate their knowledge, organise their mental actions. Moreover, many educators and psychologists argue that in conditions of the developing instruction the forming of means of intellectual activity is not the secondary, but rather one of the central aims of teaching".

Soviet and foreign researchers have developed many different systems of means of mental activities. However, methods of the forming of these means are almost not developed. In our view, the mathematics teacher should be able to form means of thinking activity on any material of a mathematical course. It would be useful to take into account the results of E. N. Kabanova-Meller (1968) who distinguishes three stages in the forming of the thinking activity: 1) forming the means in separate disciplines independently of each other; 2) forming the means on the basis of two related subjects; 3) transferring to the wide range of different kinds of activity.

There are two major directions in teaching to use the means of the thinking activity. The researchers supporting the first direction argue that it is possible to carry out the teaching to use the means of the thinking activity on the basis of the unconscious orientation in those means through exercises without preliminary generalisation of the composition of the means. This direction was developed by Covington et al. (1974), M. Rubinshtein (1980), Yu. N. Kulyutkin (1970). The latter's approach is the following: the teacher explains, stage by stage, the essentials of the solution of one version of a problem, applying all the necessary means of the thinking activity, not giving, however, the generalised description of the means. After that, pupils solve the second and the third versions of that problem with decreasing assistance from the teacher in the applying of means of the thinking activity. Further, pupils get the fourth (fifth, sixth) versions of the problem as the tests for independent solving. The success in solving of the test problem is regarded as the transfer of means of the thinking activity learned by solving previous versions of the problem.

S. L. Rubinshtein is one of the most important researchers representing the second direction which investigates possibilities of awareness in mastering means of mental activity. He wrote:

"Now two radically different conceptions of thinking confront each other... According to one of these conceptions, the thinking is mainly operating with generalisations obtained in a ready form, the intellectual activity is the functioning of the operations automatically switched on with the help of certain rules given beforehand. The problem of thinking is reduced to a problem of learning, of solidly mastering knowledge presented to a pupil in a ready form as a result of elaborating of the educational material by the teacher; thus, thinking is only the problem of a teacher, not of a pupil... The second conception emphasises the study of the process of thinking; moreover, the process of thinking is studied not where and when it operates with ready generalisations, but also... when it by the analysis of relations between and within objects and by new synthesis of the elements chosen by the analysis, leads to new generalisations" (Rubinshtein, 1976).

The pupil's awareness of the content of operations learned in many cases increases effectiveness of learning, and it is possible to conjecture that it is also true for teaching to use means of mental activity. Therefore, we follow the second conception which proposes that pupils should learn means of mental activity on the basis of the awareness of their structure. The pupils' *metacognitive* skills, their awareness of their learning, of the purposes of mathematical activities are being intensively studied now (Bell et al, 1997; Kholodnaya, 1997; Gelfman et al, 1997).

Worth mentioning is also P. Galperin's *theory of forming mental actions stage-by-stage* (Galperin, 1966). Galperin distinguishes three types of *orientation in the environment and in the object of action*. The first two types correspond to acting unaware of the content and structure of operations, but the second type is characterised by the ability to transfer actions to new objects. The third type corresponds to acting aware of the content and structure of operations.

3. The role of motivation.

Galperin and his disciples tried to make the forming of mental actions completely controllable (Galperin, 1965; Talyzina, 1984). Galperin (1966) indicated also to the importance of *motivation* for successful learning. However, he admitted the role of only "*inner* motivation of learning, adequate to the main aim of learning" (Ibid.). He wrote: "...External motivation distorts the problem itself." To the *external* motivation Galperin refers, in particular, liveliness and emotionality of the discourse, desires and needs of a pupil.

However, modern psychology admits the role of *personal* motivation of learning, i. e. motivation based on personal aims, needs and values (Leontyev, 1983). The role of personal values in acquiring any knowledge (by scientists as well as by students) was emphasised by Polanyi (1960).

Soviet psychologist O. K. Tihomirov (1981) distinguishes *informational* and *psychological* theories of thinking:

"Real objects, or named objects, participating in the statement of a problem, have such important characteristics as value; operations with these objects, i. e.

transformations of a problem situation also have different values... Formal representations of the statement of a problem (for example, as the graph or series of signs), reflecting some reality, at the same time digress from such objective (given to the subject) characteristics of the statement of a task, as proportion between various values of the elements, on the one hand, and ways of transformation of a situation, on the other hand, as the intention of the composer of a problem. These characteristics, lost in formal representation... not only really exist, but also determine (sometimes as the main factor) the activity of the solving a task... The taking or not taking into account of this fact distinguishes the psychological and informational theories of thinking".

As the examples of personal motivations one can take cases when people successfully solve mathematical and logical problems in the context of their professional (or in other way habitual) activities.

When teaching mathematical logic for prospective physics teachers, one of us encountered the following phenomenon: some male students, not able to solve problems on switching circuits using logical rules, easily solved them using their technical, engineering intuition because they had experience of constructing such circuits. Thus, the logical reasoning was not absent in those students but it could be revealed not in a direct way (demanding formal logical solutions of problems), but rather using their practical experience and activity.

4. Main characteristic traits of mathematical thinking.

In order to properly develop mathematical thinking, it is necessary to know main traits of such thinking.

Krutetsky (1968) distinguishes the following features of abilities to mathematical thinking: 1) Abilities for formalised perception of a mathematical material, for seizing the formal structure of a task. 2) Abilities for logical thinking in the sphere of quantitative and spatial relations, ability to think by mathematical numerals. 3) Abilities to reduce (to shorten) the process of mathematical reasoning and the system of the appropriate operations. 4) Reversibility of the thinking process during the mathematical reasoning.

For the development of mathematical thinking the clearness of a statement of a problem or a task is important. This clearness should come from the teacher and must be transmitted to the pupil. In particular, the clearness in the formulation of questions is important. The content of a question should be absolutely clear. For example, after the reading of the text of a task the large majority of the teachers asks the pupils: "Did you have understood the statement of a task? " or " Is everything clear to you?" This is only due to the tradition, because neither teachers, nor pupils can provide the answer to this question. It would be better to pose questions of the type: "What is given in the statement of a task? What is required to find? From what it is necessary to begin a solution? Have you ever solved a similar task? " etc. The question should be asked in such way that it could suggest the direction in which one should search for the answer, indicate what is necessary to find.

It is necessary to correctly determine *the dose of the help* (for example, through an appropriate question). It is important in problem solving, especially during the searching for different solutions. In determination of such doze of the help, in the choice of appropriate questions the determination of the *zone of proximal development* of a pupil (Vygotsky, 1996, p. 344) can help.

Determination of the zone of proximal development of a pupil is important also for the correct construction of such system of tasks that would be, on the one hand, not too difficult to understand and to solve and, on the other hand, would promote development of creative thinking, not requiring only routine reproductive activities.

For constructing such system of tasks, it would be useful to apply elements of the *method of expedient tasks* of S. Shohor-Trotsky (1915). This method was partly used in creating mathematical programs for elementary school in 70-s (Moro, Pyshkalo, 1978).

5. Understanding.

We are criticising the typical question of the teacher: "Is everything clear?" Nevertheless, without understanding by pupils of what is being spoken about in the classroom, it would be difficult for them to learn the educational material, to say nothing about their appreciation and interest.

The term "to understand" could be described in the following way: to understand a phenomenon means to uncover the essence, to reveal the reasons of its origin, its correlation with other phenomena, its place in a system of surrounding phenomena. The act of understanding can not be momentary, it has a variety of the interconnected parameters.

What concrete components of this concept would work in teaching mathematics at school and might be used in elaborating various teaching methods?

1) No mathematical object can be correctly understood, if it is considered in isolation, without its connections with other objects. The practice shows, that when this principle is ignored, the understanding of a material is often impossible. It happened, e. g., when the level of abstraction in teaching the first course of elementary geometry was raised too high, and when such add-ons as vectorial algebra or elements of calculus were suddenly introduced into school programs, when the themes of courses appeared to be poorly or insufficiently connected to other material. If this problem is now in some extent solved for the scientific (subject matter) content of the educational material (to say the truth, here also there are many problems yet), it is completely not elaborated in relation to the ways of actions used (means of mental, teaching and learning activities).

2) It is very important to teach the pupil to deduce some corollaries and conclusions of the fact studied (number of such corollaries, the level of their

significance and complexity depends on individual abilities and singularities of the pupils). It is the process of deriving such conclusions that ensures the understanding of the fact itself. Note that many aims of mathematical education also are connected with the development of the skill to make conclusions.

3) Pototsky (1975) wrote: "To understand a mathematical theory means to realise in which way it has arisen from previous results, how it is connected with them, what its origin is". This leads to the use of the *genetic principle in teaching.*

6. The genetic approach to mathematics teaching.

The *genetic principle* in teaching mathematical disciplines requires that the methods of teaching a subject should be based, when possible, on natural paths and methods of cognition inherent to the corresponding branch of science. The teaching should follow paths of the origin of knowledge.

However, one should not reduce the genetic approach to merely historical.

Wittenberg (1968, pp. 127-128) wrote:

"What is the principle of genetic instruction? Let us say in anticipation that it does not consist in retracing the historical development of a given discipline; we may consider this old mistake as done away with. In genetic teaching the historical development may be retraced in some instances, but this is done because it is the relevant and not merely the historical approach. Thus the genetic principle is a principle of relevance. It consists in presenting the subject-matter as developing out of the principles that have determined its present form.

To teach genetically thus means to base the teaching on the substantial principles of the subject-matter. It is not a question of didactical motives... but rather of principles that penetrate to the essence of the subject and make clear why the concept formations and theorising have developed in the way they did".

Many years ago the original and deep understanding of the genetic principle (not reduced to the historical approach) had been shown by N. A. Izvolsky (1924):

"In the usual course of teaching neither the text-book, nor the teacher do not make anything in order to answer (in some form) the question about the origin of the theorems. Only in rare instances we see exceptions: some teachers in this or that form pay their attention to the origin of the theorems; for the pupils of this teacher the geometry course accepts other character and ceases to be the mere set of the theorems. Moreover, sometimes some of the pupils, independently of both a text-book and the teacher, half-consciously come to the idea that a theorem has appeared not because of the wish of the author of a text-book or the teacher, but rather because it gives the answer to the problem that has naturally arisen during the previous work... Perhaps this idea of the development of the content of geometry does not reflect to a great extent the historical path of this development, but this view is the answer to the naturally arising question: how the development of the content of geometry could be explained? For the teaching of geometry to have such view of the subject-matter is extremely valuable..." (p. 8).

Izvolsky expresses the essence of the genetic approach by the following sentence: "A view of geometry as a system of investigations aiming at finding answers to the consequently arising questions" (p. 9).

7. Conclusions.

In mathematical education not only learning mathematical concepts by pupils, but also the development of their thinking abilities, mastering the means of mental activities are important. For this purpose, the appropriate motivation, fostering the characteristic traits of mathematical thinking, recognising the patterns of understanding, the use of the genetic approach to the mathematics teaching might be helpful.

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THE ROLE OF FIGURES IN GEOMETRICAL PROOF-PROBLEM SOLVING
– STUDENTS' COGNITIONS OF GEOMETRICAL FIGURES IN FRANCE AND JAPAN –

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Abstract: The purpose of this research is to clarify characteristics of students' cognitions of figures and inferences underlying their 'cognitions of figures' through investigations in France and Japan. In each country, the subjects are six students in secondary school students. The main conclusions are as follows:(1) French students recognized 'generality of triangle', whereas several Japanese students did not recognize it and they could not produce the proof in the problem. (2) In each country, 'operative apprehensions' were related to 'perceptual apprehensions' or 'discursive apprehensions'. (3) In each country, 'perceptual apprehensions' were produced by non valid anticipations of 'procedures for proof'. The 'discursive apprehensions' were produced by inferences based on 'definitions' or valid 'procedures for proof'.

1. Introduction

This paper is a report of France and Japan Cross-Cultural Research on the Role of Figures in Geometry(*) since 1997. We have interested in the geometrical proof-problem solving in secondary schools in each country. We have clarified that geometrical figures are important mathematical objects that help students in producing their conjectural activities in order to solve the problems (Harada, et al., 1993).

However, we have observed that the geometrical figures exerted negative influences on their proof-problem solving, for example, 'visual judgments for figures' and 'lack of dynamic viewpoints of figures' (Harada, et al., 1993) and 'depending on prototype example of triangle' (Gallou-Dumiel, et al., 1997).

From this point of view, many researchers have also pointed out that the negative influences of the perceptual aspects of figure and lack of ability to see a figure from various viewpoints (Presmeg, 1986; Hershkowitz, 1990; Yerushalmy & Chazan, 1990).

In this research, to clarify the negative influences of geometrical figures in detail, we focus on 'students' cognitions of figures' underlying their viewpoints of figure. Because, in our researches, we have observed the fact that the students produce their conjectural activities based on their 'cognitions of figure' even if their cognitions are not valid.

The purpose of this research is to clarify characteristics of students' cognitions of figures and inferences underlying their cognitions of figures through the investigations in each country.

2. Theoretical Foundation of Research

As a theoretical foundation in this research, we employed "types of apprehension of figure" ("les types d'apprêhension des figures") which were identified by Mesquita(Mesquita, 1989). Because we think that it is more useful for us to observe characteristics of students' cognitions of geometrical figures and their inferences underlying operational activities for geometrical figures.

Mesquita insists the characteristics of four types of apprehensions of figures as follows:

(1) Perceptual apprehension(l'apprêhension perceptive): it is a type of apprehension based on perceived properties of figures and relates to the figural organization laws in Gestalt Psychology.

(2) Operative apprehension(l'apprêhension opêratoire): it is a type of apprehension based on modifications or transformations of a figure. A heuristic function which gives students an insight to a solution of problem will be produced by this type of apprehension.

(3) Sequential apprehension(l'apprêhension sêquentielle): it is a type of apprehension based on construction sequences of figure. The constructions of figure are necessary for this type of apprehension.

(4) Discursive apprehension(l'apprêhension discursive): it is a type of apprehension based on the hypothesis in the problem. The hypothetical-deductive proof will be produced by this type of apprehension.

We especially focused on the "operative apprehension" in this research because there are various ways of apprehension of figure and heuristic functions to find solutions of the problem in the operative apprehensions.

Mesquita describes that there are four kinds of "modifications of configurations(modifications configurales)" as the ways of the "operative apprehensions".

(a) Mereologic modification (modification mêreologique): it is a modification which we can divide a whole figure into sub-figures and integrate those sub-figures.

(b) Ensemblic modification (modification ensembliste): it is a modification which we can divide elements of figure (line, circle, etc.) into several elements or take out the elements from the figure.

(c) Optic modification (modification optiqe): it is a modification which we can modify a figure from a specific viewpoint, for example, larger or narrower, and horizontal or slant.

(d) Positional modification (modification positionnelle): it is a modification which we can modify the positions or the orientations in the figure.

3. Methods of Investigation

(1) Problem of Investigation

ABC is triangle. Triangle PAB and triangle QAC are equilateral (P and Q are the points outside of triangle ABC). Prove PC=BQ.

This problem is located in common with Secondary School Curriculum in each country. The problem is given in Lycée $1^{ère}$s in France and Junior High School, the second grade, in Japan. It is a typical problem in geometrical proof-problem in each country.

(2) Subjects

In France, six students in Lycée $1^{ère}$s. They are the 11th grader students and 16-17 years old. We named them as KB, GL, NG, IL, ZR, and TA.

In Japan, six students in Junior High School, the 3rd grade. They are the 9th grader students and 14-15 years old. We named them as HK, KM, YA, NK, MY, and KD.

(3) Methods

We firstly directed students to construct a geometrical figure which was expressed in the problem and secondly to prove the proposition of conclusion. They could spend about 40 minutes to solve the problem. After they finished their problem solving, we interviewed six students in each country about their thinking processes.

(4) Viewpoints of Investigations

We set up three viewpoints of investigations for the purpose of this research as follows.

(1) What kinds of triangle ABC can students draw and what is the reason why they draw the figures ?

(2) What kinds of apprehensions of figures can they produce based on the figures and what are characteristics of inferences underlying their apprehensions of figures ?

(3) What are characteristics of students' apprehensions of figures and inferences underlying their cognitions of figures from viewpoints of France and Japan cross-cultural research ?

4. Results of Investigations and Considerations in France

(1) Representations of arbitrary triangle ABC and the reasons

(a) Students NG and IL drew acute-angled triangles (horizontal base).

(b) Students KB, GL, ZR and AT drew acute-angled triangles (non horizontal base).

(c) There was no student who drew special triangles (isosceles or equilateral).

The students ZR and TA could produce the proof. The students KB and GL could find valid procedures for proof but not produce the proof. The students NG and IL could not valid procedures for proof. However, in interview, KB, GL, NG and IL could remember theorems correctly and say the properties of theorems, for example, the conservation of lengths.

In interviews for them, we also observed the following fact. It is not important for them whether the base of triangle ABC is horizontal or not. For them, a triangle which is not an acute-angled is not a good representation for an arbitrary triangle. They are afraid if they draw particular triangle to see properties from consequences of this particularity. We think that they prefer acute-angled triangle because usually teachers draw this type of triangle.

(2) Apprehensions of figure

(a) The operative apprehensions were related to the perceptual apprehensions.

We observed that, the first step of their problem solving, 'operative apprehensions' were related to 'perceptual apprehension'. For example, Student IL apprehended that triangles PQC and PAI are homothetic as a center P because she apprehended perceptively that two lines AI and QC are parallel.

(b) The operative apprehensions were related to two types of discursive apprehensions.

We observed firstly that 'operative apprehensions' were related to 'discursive apprehensions' based on 'definitions'. For example, student NG apprehended the displacements of points on the sub-figures (e.g. triangle ABQ) and expressed the displacements by using vectors $\vec{QB}=\vec{QA}+\vec{AB}$(the definition of addition).

We observed secondly that 'operative apprehensions' were related to 'discursive apprehensions' based on 'procedure for proof'. For example, Student TA apprehended that triangle APB correspond with the triangle ABQ by using the rotation of center A and angle of $\pi/3$(procedure for proof).

(c) The operative apprehensions were related to three types of modifications of configuration.

We observed that 'operative apprehensions' were related to 'mereologic modification' (students GL, IL and AT), 'optic modification' (student IL), and 'positional modification' (students KB, GL, NG, ZR and TA).

— The students GL, IL and TA apprehended some sub-figures (triangles) by dividing a figure (mereologic modification).

— The student IL apprehended the homothetic triangles PAI and PQC as a center P as well as triangles QAJ and QPB as a center Q (mereologic modification and optic modification).

— The students KB and ZR apprehended a rotation of 'points' as a center A and angle of $\pi/3$. (positional modification)

— The student TA also apprehended a rotation of 'triangle' as a center A and angle of $\pi/3$. (positional modification)

— The students GL and TA apprehended the displacements of points from the viewpoint of vectors. (positional modification)

(3) Characteristics of inferences underlying apprehensions of figure

We consider characteristics of inferences underlying 'perceptual apprehensions' and 'discursive apprehensions' of figure which affect on 'operative apprehensions'.

(a) Perceptual apprehensions involve non valid anticipations of 'procedures for proof".

As mentioned above, the student IL apprehended that triangles PAI and PQC were homothetic based on perceptual judgement, AC//PB and QC//AB. We think that she produced "perceptual apprehensions" based on non valid anticipation of "procedure for proof". The anticipation was determined by perceptively observed outcomes and related to particular contexts. The anticipation was "local" (Piaget & Garcia, 1987, p. 110).

(b) The discursive apprehensions involve inferences which are driven from "definition" or valid "procedures for proof".

The students NG and IL produced the "discursive apprehensions" based on "definition". For example, student NG produced the displacements of points on sub-figures and expressed the displacements of points by using vectors, "definition of addition". We think that she produced the apprehensions based on inferences which were driven from "definition". The inferences occupied a part of total system of inferences (proof). The inferences were "systemic" (ibid., p. 110).

The students ZR and TA produced the "discursive apprehensions" based on valid "procedures for proof". For example, student ZR produced the displacements of points by using rotation of center A and angle of $\pi/3$. Using the properties of rotations, a proof was completed. In this case, the inferences provided reasons for the observed general facts and constructed a total system of inferences (proof). The inferences were "structural" (ibid., p. 110).

5. Results of Investigations and Considerations in Japan

(1) Representations of arbitrary triangle ABC and the reasons

(a) Students MY, KD, KM, and HK drew acute-angled triangles (horizontal base).

(b) Student YA drew an isosceles triangle (horizontal base).

(c) Student Nk drew an equilateral triangle (horizontal base).

The two students MY and HK who drew acute-angled triangles were aware of an "universal proposition" about the triangle. The two students KD and KM who drew acute-angled triangles were aware of "non particular proposition" about the triangle. The students YA and NK who drew special triangles produced a particular proposition based on compositions of inferences. That is, "there were not any conditions of triangle." ⇒ "we can draw any triangles." and "we can draw any triangles." ⇒ "we can draw the special triangle." Then "there were not any conditions of triangle." ⇒ "we can draw the special triangle."

(2) Apprehensions of figure

(a) The operative apprehension was related to the perceptual apprehension.

We observed that, the first step of their problem solving, "operative apprehensions" were related to "perceptual apprehension". For example, student KM said, "I picked up some pairs of triangles, for example, △PCB and △QCB." Student HK said, "I picked up some pair of triangles, △POB and QOC, and △PBC and △QBC." We also observed that the "perceptual apprehensions" were related to "prototypical example" of triangle: △PBC and △QBC (students HK, KM, YA, and NK), or two triangles which do not overlap each other: △POB and △QOC (student HK).

(b) The operative apprehension was related to two types of discursive apprehensions.

We firstly observed that "operative apprehensions" were related to the "discursive apprehensions" based on "definitions". For example, student NK picked up two triangles, △APC and △ABQ. But she could lead only propositions AP=AB and AQ=AC based on "definition" (the definition of equilateral triangle).

We secondly observed that "operative apprehensions" were related to the "discursive apprehension" based on "procedures for proof". For example, students HK, KM, MY, and KD could prove that two triangles APC and ABQ are congruent, so they apprehended those congruent triangles based on "procedures for proof".

(c) There was a shift from perceptual apprehensions to discursive apprehensions in the problem solving.

We observed that, the first step of their problem solving, "operative apprehensions" were related to "perceptual apprehension". However, testing the validity of the "perceptual apprehension", the "operative apprehension" were related to the "discursive apprehension". For example, student KM said, "I picked up some pairs of triangles, for example, △PCB and △QCB." - "Why didn't you make a proof by using such triangles ?" - "Because I did not find equal parts in each triangle." Next, she picked up the two triangles ACP and AQB and could prove that the two triangles are congruent.

(d) The operative apprehension were related to mereologic modifications.

We observed that "operative apprehensions" were related to "mereologic modifications" in "modifications of configuration" and not related to the other modification, for example, "positional modification" like rotation of figures. Because the students could anticipate conditions of congruent triangles as a "procedure for proof" in the first step in their problem solving.

(3) Characteristics of inferences underlying apprehensions of figures

(a) Perceptual apprehensions involve non valid anticipations of "procedures for proof".

The student HK, KM, YA and NK produced "perceptual apprehensions" based on non valid "procedures for proof". We think that they produced the apprehensions of figure based on non valid anticipations of "procedures for proof". For example, they produced a proposition that △PBC and △QBC are congruent. The triangles were "prototypical examples" of triangles. As we mentioned above, the anticipation was related to particular contexts. It was local (ibid., p. 100).

(b) The discursive apprehension involves inferences which are driven from "definitions" and valid "procedures for proof".

Firstly the student NK produced the "discursive apprehension" (AP=AB and AQ=AC in △APC and △ABQ) based on "definition" (the definition of equilateral triangle). We think that she produced the apprehension based on inferences which were driven from the definition. As we mentioned above, the inferences occupied a part of total system of inferences(proof). The inferences were 'systemic'(ibid., p. 110).

Secondly the students(HK, KM, MY and KD) produced the 'discursive apprehensions' based on valid 'procedures for proof. We think that they produced the apprehensions based on inferences which were driven from valid 'procedure for proof'. Actually, they produced the conditions of congruent triangles APC and ABQ. Using properties of congruent triangles, they completed a proof. As we mentioned above, the inferences were 'structural'(ibid., p. 110).

6. Conclusions

From our purpose of this research, we considered the results of investigations as a case study and could draw the following conclusions:

(1) Representations of arbitrary triangle

All of French students could definitely recognize 'generality of triangle' and its representation. On the other hand, several Japanese students did not recognize it and they could not produce the proof in the problem. We also will pay attention to the fact that, for students in each country, the possibility of proof depends on whether they have the recognition of 'generality of triangle' or not.

(2) Apprehensions of figure

First there were similar characteristics of apprehensions of figure between France and Japan. That is, 'operative apprehensions' were related to 'perceptual apprehensions' or 'discursive apprehensions'. There were two types of 'discursive apprehensions': based on 'definitions' and 'procedures for proof'.

Second, in Japan, there was a shift from 'perceptual apprehensions' to 'discursive apprehensions' in their problem solving processes.

Third, in France, 'operative apprehensions were related to three modifications of configuration: 'mereologic', 'optic', and 'positional'. On the other hand, in Japan, 'operative apprehensions' were related to only one modification of configuration: 'mereologic'.

(3) Characteristics of inferences underlying apprehensions of figure

In each country, the 'perceptual apprehensions' involve non valid anticipation of 'procedures for proof'. The anticipation is 'local'. The 'discursive apprehensions' involve two types of inferences: 'systemic' inferences based on 'definitions' and 'structural' inferences based on valid 'procedures for proof'.

(4) Considerations from viewpoints of cross-cultural research

French students produced flexible, creative and personal way of thinking in this problem solving, whereas Japanese students did not produce various thinking. For example, French students produced various activities in 'modifications of configuration' and 'procedures for proof', whereas Japanese students produced only one 'modification of configuration' and only one 'procedure for proof'. Japanese students also tended to depend on the 'prototypical example' of triangle in their problem solving.

We think that the difference of students' activities between France and Japan depends on not only their school mathematics curricula but also their mental cultures in each country.

From viewpoint of France and Japan cross-cultural research, to clarify the difference of students' activities between each country is a problem remained in the future.

Note

(*) France and Japan Collaborative Research of an agreement between Université de Grenoble and University of Tsukuba

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Thinking about the discursive practices of teachers and children in a 'National Numeracy Strategy' lesson.

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In this paper I reconsider the appropriateness of theoretical frameworks used in mathematics education research and argue that researchers have to confront the inevitable cultural nature of their subject of study. I outline an analytical strategy that acknowledges the complex cultural nature of mathematics education practices; a 'toolkit' drawn from Foucault's analytics of power, and show how this can be used to examine mathematics classroom discourse; in particular, an extract from the National Numeracy Strategy video guidance. I consider how this analytical strategy might be used to highlight camouflaged facets of the mathematics classroom practices; and also consider the effects of these practices on learners and teachers.

Introductory remarks

It has been noted that little significant change has been achieved in UK mathematics education practices despite major reform initiatives in 1980s & 90s of the National Curriculum and GCSE examination, particularly in terms of a persistent failure to help particular groups of learners learn school mathematics (Askew 1991, Brown et al 1998). This is paralleled in my experiences working in initial and in-service teacher development (Hardy 1996). I argue that to address this failure those of us working in Maths education research need to acknowledge that our work is deeply contextualised and cultural in every sense. Together with a growing body of maths educators (e.g. Lerman (1998), Zevenbergen (1996), Valero & Vithal (1998)) I am striving to think differently about the nature of the objects of our studies, our concepts and our methods of study. This leads to an exploration of the form of the cultural practices that are school mathematics education and a search for theoretical frames that will lead to accounts for the effects of these practices.

Liz Jones as a teacher researcher in a primary school in inner city Manchester, identified the inescapable failure produced for certain groups of children by their experience of learning mathematics (Jones & Brown 1999). Her research focussed primarily on how the condition of girls is constructed in nursery level education and she argued that even in nursery level education, understandings of gender roles were substantially developed and that many play activities served to reinforce these. She explored ways in which alternative teaching strategies might assist in eroding existing norms. However she found herself caught in a personal debate about whether it was better to engage the children in a critical education programme in which some of these oppressive norms were challenged, or whether she should embrace the 'back to basics' campaign that was then promulgated, to ensure that the children achieved threshold levels of achievement. Her conclusion was that the children concerned would end up on the bottom of the heap whatever she did. She found herself moving from an emancipatory attitude in which she sought to improve the children's lot to a more post-modernist fatalism.

My starting point in addressing such concerns is to identify ways of becoming aware of such effects of my practices as teacher and researcher. Paul Dowling (1991 p.2) voices this concern when he refers to Foucault's work, *'People know what they do; they frequently know why they do what they do; but what they don't know is what what they do does.'* (Foucault and Deleuze 1972 p208).

It has been suggested (Bourdieu & Passeron 1977) that schools serve to reproduce the existing injustices in society through practices seen as common sense in school, but which are based on the class structure present in society. It is this trick of power to masquerade as 'common sense' that leaves us unaware of the effects of our practices. This prompts the search for analytical strategies that might reveal the means through which the repeated failure referred to above is brought about. Mairead Dunne (1999) provides another example of this search. She finds a paucity of research that adequately considers social or cultural aspects in mathematics education literature and asserts the inadequacy of applications of some theories of learning in revealing the cultural nature of the maths education.

> *'Within recent work, including social constructivism, the notion of classroom culture carefully circumscribes its concern as within the confines of the classroom, sometimes as if disconnected from any external influences, for example, what the students bring with them into the classroom. This construction of 'culture' is evidently too limiting for the development of mathematics education research with a social justice concern. Such work necessarily connects the micro- to the macro- level; local practices to policy; individuals to communities.'* Dunne 1999 p117

My theoretical framework: Discursive practices

I have found that frameworks, where the actors, objects and concepts of study are seen to have been constructed through 'discourse', are particularly helpful in recognising the complex nature of mathematics education practices and in revealing hidden or camouflaged aspects of those practices. I take the term 'discourse' in its broadest sense, referring to anything communicated using signs (including actions and interactions in the classroom, resources used, and arrangements of the furniture). I will work with Foucault's development of this to examine what he calls 'discursive practices' in society. The modifier 'discursive' stresses the ways in which all practices are bound up in systems of knowledge. These 'knowledge producing' systems infuse everyday activities. Foucault argues that these discursive practices differentiate people in relation to cultural norms that, often, have become self-regulatory ways of knowing (Foucault 1972). Foregrounding the discursive nature of mathematics education practices shifts away from seeing culturally accepted norms perpetuated through traditions and sees knowledge as produced through a process of describing and ordering things in particular ways. For example, the categorisation embedded in the constitution of the National Curriculum for Mathematics in the UK (DFE 1995) - ordered into 8 levels and 4 attainment targets - became a cultural norm, regulated through descriptions that came to be taken as natural and obvious.

These theoretical notions help me become aware of the constitutive nature of mathematics education classifications and descriptions and to think about how things might be described differently.

Power is productive

When I strive to consider the effects these discursive practices have on both learners and teachers, Foucault's discussion of the productive nature of modern power proves valuable. He rejected totalising schemes that anchored power in dominant classes. He abandoned individualistic ways of viewing power which I myself had become dissatisfied with in my own research and understands power as a property of relationships - that is, not invested in one individual to exert over another. He developed perspectives that interpret power as dispersed, productive, and dynamic Organised forms of knowledge, working together with their associated institutions, have significant effects on people and their possible actions, repressing and enabling.

> *If power were never anything but repressive, if it never did anything but to say no, do you really think one would be brought to obey it? What makes power hold good, what makes it accepted, is simply that fact that it doesn't only weigh on us as a force that says no, but that it traverses and produces things, it induces pleasures, forms, knowledge; it produces discourse. It needs to be considered as a productive network, which runs through the whole social body, much more than as a negative instance whose function is repressive.* (Foucault in Rabinow 1986 p61)

Such an analysis can attend to the discursive nature of professional practices; to how simultaneously human beings (teachers and children) are defined by discourse's use, whilst at the same time the discourses describe them.

I have experienced this defining nature of discourse when, as a researcher observing in a classroom, a child has tried to elicit 'my help'. For this child a characteristic of a 'teacher' might be 'one who helps'. I will define myself as 'teacher', or not, by my response to her. At the same time I contribute through that response to the definition of what a teacher is and does. I have entered this room with labels 'non-participant observer' and 'researcher' in mind. I know, however, that I have not shed my 'teacher' self as I pass through the door. I become acutely aware that my response will define who I am and what I am in this classroom context. It will also determine my possible actions in this arena.

In this analysis I can look for traces of the productive and determining effect of power, what enables actors to act, to take a particular position, and to be heard.

My current exploration: an analytical strategy

The next section is offered as an example and as a trial of the preceding theoretical discussion of analytics. I outline a particular strategy; a 'toolkit' drawn from Foucault's analytics of power and the production of knowledge, and use this to examine mathematics classroom discourse. An extract from the National Numeracy Project video (NNP1998) is used as a short exemplar for such an analysis. The National Numeracy Strategy (DfEE 1999), the latest UK major initiative, can be seen as foregrounding the teacher's role in the child's learning of mathematics. This gives a timely context for an examination of the effects of this emphasis upon the actors (teachers, learners etc) in the maths education arena.

My 'toolkit': Normalisation and Surveillance

Any mathematics education discourse positions and categorises children (and teachers) in particular ways. For example, children are often portrayed as the ones who 'have difficulties' or 'misconceptions'. There are two particular techniques of power that can bring about this pathologising - 'normalisation', and 'surveillance'.

The process of normalisation is the mechanism that categorises people into normal and abnormal. Linking the notions of normalisation and power as a productive network allows us to see the process that determines what is considered to be valid knowledge in the classroom and how that knowledge can be expressed and by whom. It is the normalisation process that determines, for example, who 'has the difficulties' and who does not. Foucault (1977, p.184) also claims that examination '... *is a normalising gaze, a surveillance that makes it possible to qualify, to classify, and to punish. It establishes over individuals a visibility through which one differentiates them and judges them.'* Specific forms of surveillance bring about normalisation e.g. individualisation and totalisation. For totalisation, a group specification is given, asserting a collective character. This is a readily recognisable element of pedagogic activity where 'we' or a class name is used to address whole groups of participants. For example, a teacher may praise a whole class by saying 'Well done, 3W. I'm pleased with the way that you moved back to your desks'. Individuals and their behaviour are ignored by this statement. It regulates the group behaviour and asserts group characteristics. A child claiming an individual voice could find herself excluded from the group and from the classroom culture. Individualisation is the technique of giving individual character to oneself and may be an attempt to resist unwelcome totalisation. However it can also be a way of drawing attention to a child's deviance from the classroom norms and establishing their abnormality – again, a common classroom practice.

Foucault invites us to use these particular 'techniques of power' to form an analytical 'toolkit' to look at how power relations function at the micro level and to reveal constructed nature of institutional practices (Foucault and Deleuze 1972). Foucault uses the word technology referring to the physical or mental act of constructing reality. So in using the term technology Foucault emphasises the way that human beings are essentially constructed by the seemingly non-discursive background practices into which they are thrown.

The video extract given below can be seen as a text produced in a particular discursive practice – an exemplar of teaching practices promoted by the National Numeracy Project team- that can be analysed at the micro level in the search for patterns of practice. I use the Foucauldian notions of Power as productive, Normalisation, Surveillance to highlight hidden facets of the classroom practices of this 'Numeracy lesson' and to give a 'reading' of the classroom interactions and teacher's descriptions of her work.

Transcript of video extract

This extract (NNP 1998) is made up of interview scenes where the teacher comments on her classroom practices. These are interspersed with classroom scenes where the teacher moves around the room asking short calculations questions to the whole class. Her questions and instructions are presented to the whole class, whether she is referring to the children as a whole group or as individuals. The children's desks are arranged in blocks of 6 and each child has 2 sets of cards, both numbered from 0 to 9, in front of them. They hold up cards to show their answers to the questions asked.

Comment from Teacher: A few children don't put their hands up. They try to hide but that's the idea, there is no hiding place. You encourage them all as long as you give them positive feedback. Even if they get it wrong they are not scared to give an answer.

In classroom scene:

Teacher: 'Show me a multiple of 5 bigger than 75…
Is that a multiple of 5 though Michael? It's bigger than 75 but check it's a multiple of 5…. Well done Sarah!

Teacher: Show me three threes …
Three threes? Check again please, Lauren. Check please, Joe. You are looking at someone else's. Don't just look at someone else's.
If you're not sure get your fingers and count in lots of 3. Let's do it together - (chanting) 3, 6, 9. You should be showing me 9 there.

Comment from Teacher: Some children don't have instant recall of 3 3's but I've given them a method to work it out. 'Get your fingers and count in threes' - so as long as they do regular counting in threes and they're got that pattern, they have got a method to do it. When I see the children struggling I take them back to the method or strategy that we're talked through together to help them through that. They are not stood in queues waiting to get a book marked; they are getting instant feedback. They are not scared to get an answer wrong. They're having a go, they are risking things and you don't gain anything unless you have a few risks and that's what they are doing.

In classroom scene:

Teacher: Have a quick check of that one, Misha. You should be showing me twelve.

Comment from Teacher: It really works. We're seen it work. The children are motivated. The children want to learn. You never have to tell children 'Are you messing around?' They're not. They are trying. They might not be succeeding but they are trying. They really love the pace. Children don't like sitting for 20, 30 minutes on one task especially if they are struggling on it. This doesn't allow that. The children have to find answers. They work together. They help each other but they are also pushing forward. The task is changing all the time. As long as you stay focussed on target, most lessons you achieve 80% of children come out learning something that they didn't go in knowing and that's a wonderful experience and encourages you to go on further.

A Foucauldian reading:

Surveillance: the normalising gaze: This is brought about through the bodily position of the each child. By holding their cards up, by putting their hands up, they expose themselves to this surveillance. 'There is no hiding place'. By holding up their answer on the cards or by offering their answer verbally they identify themselves with their answers. They as are judged as individuals by the rightness or wrongness of their answers. It is the child who is wrong not just the answer. *'Have a quick check of that one, Misha. You should be showing me twelve'.*

This jars somewhat with the teacher's comments on encouraging them all. She states that by giving them positive feedback they are not scared to give an answer even if they are wrong. *'They're having a go, they are risking things and you don't gain anything unless you have a few risks and that's what they are doing.'* In order to be included in this classroom culture the children have to be prepared to reveal themselves, to take risks, to risk being declared wrong /not normal/ excluded. It is worth asking who runs the biggest risk here. In the classroom scenes it can be seen that significantly more boys than girls are raising their cards and that they do so more rapidly. The teacher believes that she surveys them all -*'There is no hiding place'*. However it may be that some children escape her gaze, either because of the strength of her assumption that they are all willing to have a go or because it is not practicable for the teacher to look everywhere all the time. It may be that particular children can use this to 'resist' running the risk being 'wrong'.

Certainly it is not the teacher who has to take risks with right and wrong answers here. She is not seen here to follow her own adage, '*you don't gain anything unless you have a few risks'.*

Individualisation and totalisation: Looking at the classroom practice I can ask what feedback is given and what constitutes positive. There are ideas of pace, quick responses, instant feedback in terms of right and wrong, ALL are given feedback, no waiting, no struggling for a sustained length of time. Individual children are told when they are wrong albeit with some gentleness. The teacher might claim that she uses children's names to give praise and to encourage them to try again *'Three threes? Check again please, Lauren.'* The effect of this individualisation may be different. Is it positive feedback or is it obvious to Lauren that her answer is wrong? Does 'check again' soften the effect of being labelled abnormal?

This is more marked for Joe, *'Check please, Joe. You are looking at someone else's. Don't just look at someone else's'.* The knowledge constructed through these discursive practices is not just about right answers but there is a clear production of 'wrong method' and a privileging of one particular 'right method' – *'If you're not sure get your fingers and count in lots of 3. Let's do it together - (chanting) 3, 6, 9. You should be showing me 9 there'.* The totalisation of *'doing it together'* will help define this as THE valid method to use if a child cannot recall the right answer, the valid knowledge for that classroom.

Again, though, there is a possible contradiction - the teacher says, *'They work together. They help each other but they are also pushing forward'.* This may open up a way back from exclusion for Joe.

Power is productive: The teacher's is the only voice as she acts out her part. Perhaps it would be claimed that this is inevitable in a video demonstrating 'effective teaching'. There can be seen a clear description of teaching and arguable disassociation of learning from the acts of teaching. The teacher can occupy her position by meeting the description of 'effective teaching'. For example, the teaching should be pacey. It is then presumed - in fact defined - that the learning is good (or perhaps the learning is absent from the scene; replaced by short sharp activities that keep them busy and on their toes.)

There is a fracture between the given description of what constitutes effective teaching and description of a successful lesson where 80% of children learn something new – that is the recognition of the 'non-learning' of 20% of the class. A trick of totalisation hides the abnormal 20% so the teacher evades the need to question her teaching, a potentially risky business. This reveals the working of the discursive practices. Knowledge is constructed (e.g. Good maths tasks are short and change fast. Sustained effort is not desirable. Children do not learn through longer tasks. What they do is just struggle.) and meaning of teachers' and children's actions is oriented around it.

Concluding remarks

I have said that there is a need for Maths Education research to rethink its concepts and objects of study in order to become aware of why the process of learning school mathematics has the effects it does on groups of our children. I have written about my struggle to find more productive ways of examining my practices and have argued that to do so I must acknowledge the discursive nature of the field in which I operate.

I have discussed the general nature of tools or strategies that can appropriately be applied to classroom discourse. I have tried to show how one such strategy can reveal the discursive practice(s) of mathematics education and the ways this can position people within the classroom, effecting their actions and determining what each has to say to be heard. In particular I have readily recognised these 'techniques of power' in even a brief extract of mathematics pedagogic discourse.

I have worked with Foucault's vision of normative nature of assumption; that, for the vast majority of learners, school is an 'unreal' world - where they are not the 'normal'. If we, as mathematics educators and as researchers, start to rethink our curriculum and our practices, to undermine common sense ideas about children, teachers and mathematics we will gain some insight into why we are not helping many groups of learners learn mathematics. We also need to abandon normalising assumptions and ask the hard questions of why we act the way we do.

So what might such a hard, but interesting question be in Mathematics Education? I may wish, for example, to reproblematise aspects such as teachers' role. In practical terms what might questions might this involve asking?

> *I notice the persistence of a perception of 'maths as right or wrong' amongst pre-service students with whom I work. To explore what I do that contributes to this perception, I might critically examine the strategies and teaching styles I intend to dislodge this notion of maths as (always) right or wrong. However, I need to go beyond this to consider how power continues to work regardless of changes in school mathematics curriculum and teaching approaches so that this construct of 'right or wrong maths' remains. I should ask whether all pedagogic contexts are liable to this construction of 'right or wrong' knowledge. Why is it particularly difficult to*

resist this in mathematics? I should seek examples where this has been successfully resisted. And, most importantly, ask whether there are specific instances of resistance from learners to the normalising effects of 'getting maths right or wrong'.

My work here suggests that there are ways that mathematics education practices can be changed to disrupt and resist the current patterns of normalising. By thinking about alternatives, by exploiting the lack of stability of many of our professional notions, we might open up spaces from which we can counter ill-posed problems and look for sites of resistance

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Classroom Discussion on the Representation of Quantity by Fractions: Stability of Misconception and Implications to Practice

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A class in which fourth graders discussed actively on the representation of quantity by fractions was observed and analyzed. Many of them asserted when they divided a paper streamer of any length in two, if the length of the paper streamer was represented with "meter," the length of one in the two was 1/2 m. Since the knowledge system on fractions the students held was closed and complete, an assertion that 1/2 m was a half of 1 m brought no change in the system. Knowledge system should be open to advanced learning from the outset of learning. According to the line, suggestions to the teaching and learning process on the introductory part of fractions are discussed.

1. Introduction

In the field of mathematics education, the following are widely accepted: Mathematical knowledge is not transmitted from a teacher to students but is constructed by each student, it grows through "doing mathematics" such as students' discussion, inferring, guessing, and refuting in a mathematics classroom. Similar to other knowledge it is also situated and students come to their mathematics classroom with ideas and experiences on it before it is taught in the classroom.

In this paper, I report a class on fractions in the fourth grade. In Japan, proper fractions are introduced in the third grade and improper and mixed fractions are in the fourth grade. Many math textbooks for third graders treat proper fractions based on the division of a paper streamer of 1 m-long or a container of 1 ℓ. On the other hand, it is also well known that many students hold misconceptions on the representation of quantity by fractions. For example, Fig. 1 shows a typical response to a drawing problem of length represented by a fraction.

There is a paper streamer of 2 m-long. Shade 2/3 m of it.

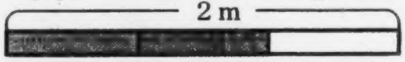

Fig. 1. Typical response to a drawing problem.

The typical response is generated from operating 2/3 to a given object (the paper streamer of 2 m-long, in this case), regarding "2/3 m" as a fractional operation. It is not a light mistake but a kind of serious misconception hardly to correct. Similar responses are observed in problems to locate fractions on a number line and reported as an error based on the "part of a whole" model (Novillis-Larson,1980; Kerslake,1986). In this paper, I take up a class of fourth graders on the representation of quantity by fractions and then refer to the cause of misconception and suggestions to the teaching-learning process on fractions. In the class, students discussed actively, through which their misconceptions were revealed exhaustively. They had learned proper fractions and a bit of decimals but not yet improper and mixed fractions. The following, the students who insisted conformed to the part of a whole model and who to the division of the unit (1 m) are mentioned PW students and UD (unit division) students respectively.

2. Observation of the class

The teacher distributed two paper streamers to each student, one was red and 1 m-long and the other was white and its length was not 1 m, of which length was various with students. Then she asked them to make a paper streamer of 1/2 m-long with the white one. Many students made a half of their white ones, which produced various lengths of "1/2 m." The following illustrates a part of the outset of the first lesson.

S1: The lengths are various. I feel it funny. The reason is 1/2 m is a half of 1m, so you must make 1 m first.
S2: I think, whatever the length, if we break it to a half, it becomes 1/2 m.

Similar assertions continued, then the teacher asked students which idea they supported. Seven students agreed to S1, 26 students to S2, and two students responded they were not sure.

S3: If they are various, they are 1/2 but not 1/2 m.
S4: When is 1/2 m decided to be a half of 1 m? (S3: You say when, but ····.)
S5: 1/2 is a fraction. It is made to show something, like 15.333, not to represent by integers, and it doesn't represent only 1/2 of 1 m. So, it is not funny (even the lengths are various).

Some students argued on what 1/2 m and 1/2 were all about, however, the discussion continued without clarifying their definitions.

S6: 1/2 m is different from 1/2 and it is added meter after 1/2, so it is a half of 1 m.
S7: Then, what do you say a half of ten? If it is 10 m, there is "meter." Don't you say 1/2 m? Tell me why you don't say 1/2 m (about one half of ten meters).

As students' discussion was deadlocked, the teacher combined two paper streamers of "1/2 m"-long, stuck it with a paper streamer of 1 m-long on the blackboard, and asked to make a problem. Some problems were proposed, from where a problem common to all: "What value do you get if you joint 1/2 m and 1/2 m?" was taken out.

S8: The equation is 1/2 m plus 1/2 m equals 1 m.
S9: If 1/2 m is a half of 1 m, the answer is 1 m but if they are various lengths of 1/2 m, it is not 1 m.

Similar assertions were repeated afterward. In the second lesson, after the teacher reviewed students' ideas in the first lesson, the discussion was started again.

S10: "Meter" is the unit used in the world, so, if 1/2 m varies, it is funny.
S11: A half of one meter must be 50 cm.

These ideas were proposed at the outset and no objection against "1/2 m+1/2 m=1 m" was asserted.

S12: If everybody agrees the answer (of "1/2 m+1/2 m") is 1 m, it is funny. 1/2 m and 1/2 m is 1 m, so 1/2 m is a half of 1 m.
S13: When you divide unclear length in half, it is 1/2 and no unit is necessary.

Since these ideas were far ahead, all students seemed to agree with them. However, if PW students accepted the equation, they had to assent 1/2 m was a half of 1 m. Therefore, "1/2 m+1/2 m=1 m" was to be denied.

S14: If you make 1/2 of 3 m, then it (1/2 m+1/2 m) is not always 1 m.
S15: A half of 3 m is 1/2, so 1/2 m isn't always equal to 50 cm.

Some ideas to support them were posed and the discussion turned to the starting point.

T (Teacher): Is the equation (1/2 m+1/2 m=1 m) correct? (Many students: Wrong!)
T: What about you in this one (1/2+1/2=1)? (Students: Sure!)
S16: So, meter is not always 1 m!
S17: "3 m" includes meter, so, meter is not always 1 m.
T: Is it possible to say that a half of 10 m is 1/2 m?
S18: It is 1/2 m and, called 5 m, too.
S19: So, 1/2 m+1/2 m is not always 1 m.
S20: I want to ask you. Do you say a half of 1 cm is 1/2 m?
S21: 1/2 cm!
S22: If you put a half of 1 cm in 1/2 cm, you put a half of 1 m in 1/2 m, don't you?
S23: Problems are different, so, the units are different, too.

Through the discussion, PW students' (mis-)propositions were elaborated. The following assertions were observed in the latter half of the second lesson.

S24: "Meter" is added to both 1 m and 3 m. But why, in the case of 3 m, can't we add meter like 1/2 m?
S25: I think we don't say 1/2 m in the case of 3 m and 4 m, because 1/2 m is a half of 1 m.

S26: What do you say when you divide 4 m in two, except 2 m and 200 cm?
S27: The case of 4 m isn't expressed with a fraction.
S28: I feel it funny. If you can't divide 4 m, you can't divide 1 m (in two), but it is 1/2 m. A half of 10 m is 1/2 m, a half of 4 m is 1/2 m and so a half of 4 m is 1/2 m, too!

Since the students had not learned the representation of quotient by fractions, she (S27) could not respond as "4/2 m" to his question (S26). His assertion against her (S28; S26 and S28 are the same students.) was persuasive, which seemed to be supported by many students. The second lesson came to an end without reaching an agreement and UD students remained a minority in the classroom. In the third lesson, since the students seemed not to be able to reach the conclusion even though they continued their discussion along the same line, the teacher showed them "the correct answer" and explained it with related matters. Although all the students including PW students listened carefully to many kinds of assertions, the posttest carried out after the lesson revealed that there was little significant improvement (Hasegawa, 1997).

3. Discussion

The students' assertions are not peculiar to the members of the classroom. I have observed the similar utterances in several mathematical classrooms employed the similar task. However, the class was prominent on the students' vivid discussion and the firmness of assertions of PW students. I describe them in detail.

The students' discussion: Their vivid and active discussion depended heavily on the teacher's position in the classroom. She kept a neutral position, acted as a chairperson and wrote their assertions on the blackboard with arranging opposite ideas. Several times she inquired distribution of ideas among them by a show of hands, counted them and tabulated on the blackboard, which amplified their cognitive motivation (Inagaki & Hatano, 1977). Moreover, not only purely cognitive motivation but also social, "partisan" motivation to attain superiority in the class (Hatano & Inagaki, 1991) seemed to contribute to the hot argument. Apart from its direction, the class was to give students the opportunity of "doing mathematics" (Lampert, 1990) in some degree.

The firmness of PW students' assertions: If there are sharply divided ideas in a natural science classroom, a decisive experiment may show which one is correct. On the other hand, in many cases of mathematics, it is difficult to devise an effective experiment. Even though it can be developed and show clearly what and where the correct answer is (in the case of probability, for example), from where the first step of mathematical pursuit must be started. I believe fundamental resolution of inconsistent ideas depends heavily on the revealing of contradiction contained her or his knowledge system and its reorganization. However, it did not occur in the PW students, since their knowledge system of fractions constituted an operationally closed and coherent system.

According to the system of PW students, the representation of quantity by a fraction, (k/n)-meter for example, consists of a pair (x m, k/n), where x m (x meters) is an object that the students conceive or confront commonly and k/n is an operator which operates on the object. The system contains addition of lengths represented with meters, to what extent the addition is generalized is uncertain though; and it is organized separately to the addition on rational numbers ("1/2+1/2=1, but 1/2 m+1/2 m is not always 1 m"). Thereby, PW group's theory-like system became to involve coherent explanations of the representation of quantity by fractions. Needless to say, the issue exists on the point of the definition of 1/n meter and it is the question, "When is 1/2 m decided to be a half of 1 m? (S4)" that had to be replied. The assertion of UD students functioned as disturbances to the system of PW students (cf. Piaget, 1985) and the compensation to them made PW group's system more complete but operationally closed.

The situation of the introductory part of fractions in a math textbook adopted when the students were third graders is as follows: "You measure the length of a paper streamer with a stick of 1 m-long and you find the length is 1 m and a fragment shorter than 1 m (Fig. 2). How long is the fragment?"

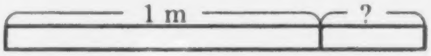

Fig. 2. An introductory problem to fractions.

Starting from such a kind of question, the explanation: "When 1 m is divided equally into three parts, one of the three is said one third of 1 m and denoted 1/3 m," for example, is introduced via students activity to find the relation between 1 m and the fragment. However, since only proper fractions were taught in the third grade, neither 1-1/3 m nor the expression "1 m and 1/3 m" is employed to represent the whole length of the paper streamer. Moreover, almost all drawings and diagrams of paper streamers, containers and number lines inserted in mathematics textbooks for third graders adopted in Japan limited their length, capacity and maximum value to 1 m, 1 ℓ and 1 respectively. These drawings are likely to promote an inappropriate generalization from "to divide 1 m" to "to divide x meters" without making aware the units. However, the generalization shows students' active construction of knowledge based on what they previously acquired.

4. Implications for classroom practice

Enrichment of ideas and activities shared among students: When there are conflicting ideas in a class, their resolution and reaching an agreement through mathematical discussion need some components commonly held by two or more groups that pose contradictory ideas. In the case I have mentioned, the idea shared by two groups, PW students and UD students, was only the equation "1/2+1/2=1," which did not enable them to resolve their opposite ideas. In different classes of fourth and fifth grades, the same task of making a paper streamer of 1/2 m-long was treated and the same responses to the task were observed at the outset of each class. The fourth graders, who did not learn improper and mixed fractions but learned decimals, agreed gradually "1/2 m+1/2 m=1 m" then "1/2 m is a half of 1 m." The fifth graders, who already learned improper and mixed fractions, could resolve the contradicted ideas smoothly when some students pointed out that 1/2 m and 50 cm were equal. It is inferred from the latter case that learning improper and mixed fractions made students realize the unit 1 m. On the other hand, it is also well known that many students find difficulty in understanding improper fractions when they learn them for the first time (Carraher,1996; Tzur,1999). These cases suggest that a classroom discussion varies according to what ideas opposite groups share. Enrichment of ideas and activities shared among students must be considered carefully before the instruction of which aim is to resolve students' conflicting ideas.

Generating of effective mental models: Even mathematical knowledge is inevitable the constraint of situation in which it is constructed. According to the recent studies that the construction of mental models serves for the desituating of cognition (Hatano & Inagaki, 1992) and that the invariant of the structure of activity plays an important role for transfer (Greeno, Smith & Moore, 1993), the mental models constructed and activities organized at the early stage of learning are significant for the long-term learning. I carried out experimental lessons to fourth graders in two classes who did not learn improper and mixed fractions. In one class, a diagram of a paper streamer of 3 m-long (Fig. 3) and in the other class a number line (Fig. 4) was adopted. When I asked to students in the former to shade (1+1/4) m of the paper streamer (Fig. 3), they proposed three drawings shaded 1-1/4 m, 1-2/4 m and 1-3/4 m, which represented 1 m+(1/4)×1 m, 1 m+(1/4)×2 m and 1 m+(1/4)×3 m respectively. A half of the class agreed on the drawing of "1-1/4 m," however, the other half replied the other two or all of them were correct and they did not reach an agreement. In the class with the number line, I asked students to mark the points: 1+1, 2+1/4, 1+1/4 and 1/4, on each of four number lines printed on their worksheets with ascertaining one by one. At that time, they agreed on each of the correct marks without great confusion.

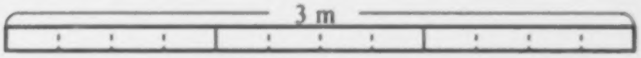

Fig. 3. A paper streamer of limited length.

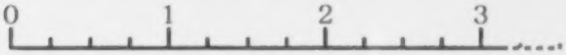

Fig. 4. A number line with suggesting its extension.

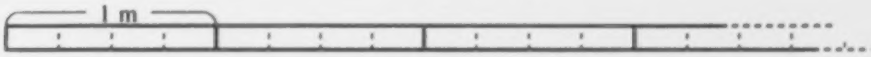

Fig. 5. A paper streamer with suggesting its extension.

These cases suggest that the concrete object with limited whole, of which whole length is 3 m in the case shown in Fig. 3 for example, causes the confusion of representation of length by fractions. On the other hand, in the case of the number line with suggesting its extension, it seems that the constraint students cannot know the whole length of the number line regulates the idea that the part of the unit interval must be considered. My another study shows that asking students to shade 1/4 m, the percentage of correct responses to a paper streamer with suggesting extension of its length and emphasizing the unit (Fig. 5) is higher than the one of which length is limited (Fig. 3). Even though the representation of length by proper fractions is dealt with at the introductory stage, the followings should be considered: To treat them with mixed and improper fractions at the same time, 1-1/3 m and 4/3 m in the case shown in Fig. 2, for example; to embed them to a longer paper streamer than 1 m and to locate their values on the number line longer than 1. These processes enable students to be aware not only the significance of the unit and proper fractions but also the integration of natural numbers, decimals and fractions.

When introducing a mathematical concept, we must avoid making students construct a closed system related to the concept. Since there are some constraints due to the first step to the concept, all matters may not be treated at the outset. However, the system must be open to advanced stages so as to compensate for its incompleteness by students' discussion and by their learning of related matters. Moreover, the more a situation is concrete the richer it contains meanings individuals hold, which may sneak into the system. Not only the variability of equally concrete situations on the concept but also embedding initially constructed concept to a higher mathematical object should be considered deliberately in the process of construction of mathematical concept.

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PROCESS OF INTERNALIZING NEW USE OF MULTIPLICATION THROUGH CLASSROOM INSTRUCTION: A CASE STUDY

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ABSTRACT

In this article, the process in which pupils acquire new use of multiplication, that is to measure areas, is analyzed. Behaviors of five 4th-grade pupils in a series of lessons in areas were studied in depth by qualitative research method. Their use of multiplication was changing as the lesson evolved, which was characterized by "using multiplication as a label," "using it positively to approach problems which have not been solved before," and "using it effectively to achieve the goal of measuring areas." These three phases show the pupils' meaning construction of multiplication in the context of measuring areas from a secondary accompaniment to a powerful tool of thinking. Concerning which phase was observed and whether there was a progress, individual differences among the pupils were notable. By comparing their behaviors, large difference in their task-interpretation activity became apparent.

Theoretical Background and Purpose of the Study

Over the last years, situated view of cognition has proposed that a person thinks not only through his/her mind but also through the assistance by and interaction with tools (Resnick et al., 1997). Notion of tool here extends the traditional view of physical artifacts to include intellectual ones such as concepts, forms of discourses and linguistic constructions. Mathematics is filled with such tools, and researchers in mathematics education have demonstrated that tools chosen or designed play critical roles in shaping one's problem solving process (e.g., Nunes, 1997; Meira, 1995). They give powerful evidence for the theoretical claim that once a person acquires the tool, it comes to structure one's thinking in significant ways (e.g., Kaput, 1991). However, we do not know much about the very process of acquisition in the midst of interaction with others. Meira illustrated in detail that student-designed tables of values were appropriated and transformed continuously in his/her interactions with the social and material settings of activity. These students were asked to solve problems after they learned the use of tables in class. Then what about the case of students who are actually learning new tool of thinking in their mathematics classroom?

The purpose of this article is to get information of the process in which pupils acquire a new tool through classroom instruction. The tool being focused is multiplication in measuring areas of rectangles and squares. It enables one to know those areas by measuring only the lengths of two sides and to know more complex areas by decomposing them into rectangles and so on. By a micro-level analysis of five pupils in a fourth-grade classroom, it is described what informal views of area they bring to class, how they meet multiplication as a new tool, and in what way their tool use evolves through interaction with tasks, peers and the teacher.

The perspective employed is that of internalization. It is the process whereby the individual, through participation in interpersonal interaction in which cultural ways of thinking are demonstrated in action, is able to appropriate them so they become transformed from being social phenomena to being part of his or her own intrapersonal mental functioning (Chang-Wells & Wells, 1993). The interest in this article is to characterize the process in the realm of pupil's tool use. Although the word "internalization" provokes an image of passive receiver, researchers generally accept positive involvement by the individual in his interpretation and construction (e.g., Nakamura, 1998). Moreover, Waschesocio (1998) argues that internalization is the key to account for the mechanism of conceptual development. Similar to them, I shed light on a person's constructions and reflections as I look through his/her use of tool (Ito-Hino, 1996; Hino, 1997). In the present article, the process is characterized by three phases in the use of multiplication to measure areas. This characterization enables an analysis of individual difference, which opens a way to talk about the mechanism of making progress in the phase.

The Lessons

In Japan, teaching of measurement of area starts in grade 4. Here, areas of square, rectangle, and compounded figures (e.g.,) are treated. In the same way as the teaching of length and bulk in earlier grades, it usually begins with the situation of comparing sizes. Sizes of different figures are compared by direct and indirect comparisons, by numbers of grids of (1cm) x (1cm), and finally by multiplication of lengths of two sides. Based on these, areas of triangle, parallelogram, trapezoid and rhombus are formulated in grade 5.

Table below is a brief summary of content in each of the total 8 lessons that the teacher in this study organized.

Day	Summary of Content
1	Introduction to measurement of area by number of squares ((1cm) by (1cm)) in a game situation, i.e., to compete the size of area one gets. In doing so, a small grid desk pad called "miracle desk pad" is distributed to the pupils. The term "area" and the unit of measure "cm^2" are also introduced.
2	Introduction to the formula of measuring square ((side) x (side)) and rectangle ((length) x (width)). Some exercises.
3	Measurement of areas in compounded figures by applying the formula. Here, different ways of measuring the areas are presented.
4	Reflection on the previous lessons. Exercises.
5	Introduction to a larger unit of measure (m^2). A paper of the size $1m^2$ is shown on the blackboard and the pupils check the relationship "$1\ m^2 = 10000\ cm^2$" with the teacher.
6	Introduction to the units of acre (a) and hectare (ha). A table that shows the relationship among "m^2," "a," and "ha" is also introduced.
7	Introduction to the unit of "km^2." The relationship among the units of measure is shown by a table.
8	Summary of the previous lessons. Chapter exercises.

Three observations of the teacher in treating multiplication are noted. First, the teacher introduced multiplication on Day 2 after pupils recognized the idea of measuring areas by counting the number of grids. On Day 2, the teacher and the pupils first discussed a rectangle drawn on the grid paper. She intended to make the pupils aware of an easier way of finding the number. In this regard, she was successful because several pupils verbalized, "Multiplication is easier," in response to her saying, "Counting would be the easiest way…" The pupils developed the formula of area of square and rectangle with the teacher. Second, on Days 3 and 4, compounded figures were treated. The teacher emphasized different ways of finding the areas such as "decomposition" and "removal." She tried to help the pupils who had difficulty by saying, "Try to use the formula we learned yesterday." Third, different units of measure were introduced successively to pupils. She repeated that the idea is the same as that of "cm^2." Overall, the teacher intended to make the pupils aware the idea of multiplication as the indirect and abbreviated measurement of area.

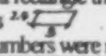
Data Collection and Analysis

In order to get a thick description of individual pupils, qualitative research method was used.

A graduate student and I started to visit a fourth-grade classroom with 29 pupils about one month before the class went into chapter "areas" (Nakahara et al, 1996). During the time the class covered the chapter, we videotaped the entire classes and took notes on the pupils' behaviors and voices. The pupils' worksheets and tests were also collected. The teacher used worksheets of her own making throughout the lessons. It has a space for the pupils to write down their own work for the task. It also has spaces so that they can write their opinions to the solutions presented by their peers, important points, and their thoughts of today's lesson. The worksheets turned out to offer a lot of information of the pupils' learning during the instruction.

Observation of five target pupils constituted an important part of data collection. These pupils were chosen based on performance of an assessment on areas conducted before the instruction. They represented a range of scores from low, middle, and high performance group. They were also cooperative in the study. They were willing to take time to be interviewed and open to communicate with the observers.

Table below shows their responses to the problems in the pre-assessment.

	Which is larger, A(□) or B(◇), and why?	Draw a rectangle that has the same area as  (the numbers were not given.)	Tell me what "m^2" means to you
Sen (Male)	B (strategy used: multiplication of lengths of two sides)	Draw a rectangle (2.4 by 3)	"Area"
Yuka (Female)	B (cut and paste the paper)	Draw a rectangle (2.4 by 3)	"Some size"
Naoki (Male)	Same (Addition of the lengths of four sides)	Draw a rectangle (2.4 by 3)	"Area?, I'm not sure"
Owa (Female)	B (multiplication of lengths of two sides)	Draw a rectangle (2.4 by 3)	"Size"
Kenji (Male)	B (compare the lengths of diagonals)	No response	"I don't know"

Before instruction, the five pupils used different ways to compare the areas of rectangle A (4x2) and square B (3x3). Only Sen knew the multiplication of lengths of two sides as a way of measuring the area. Owa used multiplication as well, but she said, "I just have heard someone said so." Four of the pupils drew a rectangle simply by using the lengths of two sides of the parallelogram. As for the symbol "m^2," pupils except for Kenji replied that they had seen the symbol and that it designates something about size.

In each session, each of us observed two pupils, whose seats were located close to each other. The other pupil was videotaped. We recorded the pupils' work, their local conversations with their peers and the teacher, and their reactions and behaviors in the whole-class activities. Interviews with the pupils were also conducted twice during the instructional period. In the interview, they were asked to solve several problems on finding areas. They also talked about the work done in class.

In the analysis, the data collected were aggregated, compared, and clustered (Lecompte & Preissle, 1993). Especially, by noticing when, where and how the pupils used multiplication, consistencies and inconsistencies were sought and unitized as categories. These categories served as the basis for formulating the emergence of senses and significance the pupils developed toward multiplication in measuring areas.

Three Phases of Internalization Seen in Yuka's Use of Multiplication

In this section, the sense and significance emerged in multiplication is characterized by three phases. In doing so, a target student's behavior in the use of multiplication is illustrated.

The first phase – Multiplication as a label

As described earlier, the pupils accessed to multiplication rather naturally on Day 2. In response to the teacher's question, Yuka, a target student, also raised her hand vigorously. She was named by the teacher and replied, "[Multiply] the number of the vertical grids and the number of the horizontal grids." She also positively wrote down "4x3" and "3x3" for exercise problems.

One of Day 2's tasks was to draw square and rectangle that has the area of "16 cm^2" on a grid paper. It was not easy for many pupils, and different approaches were seen. Yuka's thinking of this task was complicated. First, she drew a square of 8 by 8. But soon after that, she erased it by saying, "Eight times 8 is 64 (not 16)." Then she drew a rectangle of 4 by 2 and said, "It's 16 cm^2." Here, she wrote "8 cm" and "2 cm" beside the length and the width of the rectangle. Later, the teacher came to Yuka and pointed out that one grid is (1cm) by (1cm), but Yuka said, "There are 16 of these. I counted them." After that, Yuka erased it and drew a rectangle of 2 by 4, and another one of 8 by 2. Finally, she added a figure (see Figure 1) on the grid paper.

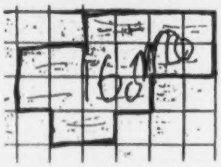


Figure 1. A figure added

Just after Yuka knew multiplication as a way of measuring area, she was observed to begin to use it positively. She quickly learned that if two numbers (length and width) are given, multiplication of the two gives the answer (area). However, as her thinking of the drawing task shows, her use was idiosyncratic. Yuka thought about two numbers 8 and 2, but could not represent them as the length and the width of a rectangle on the grid paper. She ignored the size of one grid. She seems to have believed that the rectangles she drew have the area of 16 cm^2 because they are in any way composed of 16, a little bit smaller, grids. The last figure (Figure 1) she drew is neither square nor rectangle. Here, she completely forgot to use multiplication and instead relied on counting the grids. From these observations, it can be said that for Yuka, using multiplication was secondary. She used it mainly as a label which shows others that she was doing fine in the study of area (see also Hino, 1997). For her, not the multiplication but the counting was the primary tool of thinking.

The second phase – Positive use of multiplication to approach problems which have not been solved before

On Days 3 and 4, different types of compounded figures were given to the pupils successively. The pupils were asked to measure the areas by applying the formula learned on Day 2. For the first problems (Figure 2), Yuka could not proceed further than noticing the lengths of sides. She measured every side, multiplied and divided them blindly, and got

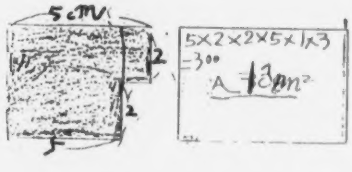


Figure 2. Yuka's work on Day 2

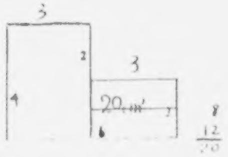
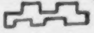


Figure 3. Yuka's work in the interview session after Day 3

all-incorrect. Later in the whole-class discussion, Yuka was observed to listen to the solutions by her peers and took notes on them. From her worksheet, she seems to become interested in the "removal" way as a new way of measuring area. This is shown by her writing of "for bumpy figures, take away a part from the whole," and "how about this () ? Is this the same?" in the worksheet. On the other hand, to the "decomposition," she was not paying attention very much. Indeed, in the interview session after Day 3, she was still not able to find the area of L-shaped figure (see Figure 3).

On Day 4, the first problem was to measure the area of U-shaped figure. Yuka got the correct answer by applying the "removal." Later in the whole-class discussion, she knew that decomposing it into smaller rectangles could measure the area of the same figure. For the next problem, a T-shaped figure (Figure 4), she worked hard to try to apply the "decomposition" as shown in the protocol below:

> *Since this whole is 6, it will be 6x3... so it's 18, and 18 plus, um, mm, here is 2 cm, here, since this is 2, and this is 5, so, um, 2 by 5 is, umm, 18, add to it, then, it would be 28, it should be (emphasizing), ..., I got the answer but, the problem is how to write the (mathematical) expression ...., (after a while) I got it! I got it! 3x6 plus 5x2 equals 28, so it is 28. (she began to hum a tune).*

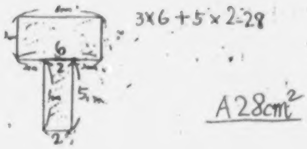


Figure 4. Yuka's work on a T-shaped figure on Day 4

As far as I know, this was the first time that Yuka spontaneously used the "decomposition" successfully. This problem could have been solved by the "removal" which she had been using. Therefore, she can be said to have chosen the "decomposition" by herself and successfully completed the solution. After this, Yuka also had an opportunity to teach this "decomposition" to her neighbor who got confused between the ways of "removal" and "decomposition." She repeated the procedure step by step four times because he did not understand easily. All these experiences would have enabled her to recognize the significance of multiplication to measure areas that had not been solved.

Learning of different units of area also contributed to Yuka's familiarizing of the use of multiplication. On Day 5, she wrote in her worksheet, "It's easy because we again need to do multiplication even in the case of m^2. I enjoy it." She naturally accepted the unit "m^2" in connection with the unit "cm^2." Here again, she recognized multiplication as working for a wider range of problems beyond the unit "cm^2."

The third phase – Using multiplication effectively to achieve the goal of measuring area

Through everyday teaching and practice, the pupils in the class got accustomed to using multiplication. However, they did not necessarily come to act in the same way. In the case of Yuka, she clearly came to show her disposition toward "decomposition." In the interview session conducted after Day 7, being asked which way she used more often, she replied, "I often use the decomposing one. The reason is because I am not good at subtraction, and I am really good at addition." Indeed, she found the area of rectangle on a grid paper by 12+12+24 (Figure 5), and explained her reason as follows:

> *... because, I don't know this beyond 24. This is 12, this is 12, 12 plus 12. I know that because it is the calculation of clock. But, for the larger one, I don't know at all (giggling), so I decompose this here, and next, here, and then, 12 is left here, so I added them all.*

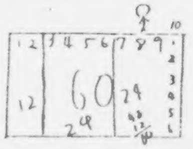


Figure 5. Yuka's work on a rectangle after Day 7

Here it should be noted that she was developing her own way of using the "decomposition" in order to overcome her weak point, that is to calculate large numbers. Here, a way of using multiplication was chosen by her on purpose to achieve the goal of measuring area efficiently and reliably.

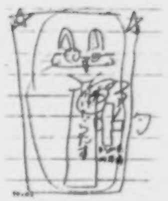


Figure 6. A cartoon by Yuka

On Day 7, Yuka began to draw cartoons in her worksheet. She was good at drawing pictures. She usually added balloons to them. These unique activities seem to have special meaning to her in objectifying her implicit use of multiplication. For example, she wrote, "you multiply" (Day 7) and "add after you break up" (Day 8, see Figure 6). In the latter, it was the name of the "decomposition" that was specially assigned by Yuka.

Individual Differences in the Appearance of the Phases

In the section above, Yuka's behavior was focused because her use of multiplication was changing rather clearly and it was easy to illustrate the different phases. However, same analysis on the other pupils revealed that there were notable differences among them concerning which phases were observed. Yuka and Naoki's use of multiplication evolved from "as a label" to "effective use." In the case of Owa and Sen, they did not show the "label" use clearly. They were more advanced, contriving different ways by themselves and choosing suitable one according to the shape in the problem. On the contrary, Kenji did not make progress beyond using multiplication as a label. He had been observed to measure the length of every side of a figure and to multiply some of them. For him, to multiply two numbers was the sole procedure to show that he was doing area.

It is curious why these differences occurred despite the fact that all the pupils were taught in the same classroom by the same teacher. The difference would partly reflect on their conceptual footing they brought to the instruction. For example, a major reason for Kenji's poor tool use would be his lack of understanding of area as two-dimensional measure. However, rather than reducing the account only to the psychological constructs, it seems more productive to look closely at their tool-use activity. Here, two examples of behaviors observed by Yuka, Owa, and Kenji are illustrated. Both show that to share and achieve the goal of measuring area by utilizing multiplication is another key point to explain the difference.

Different behaviors of Yuka and Owa on the task "draw square and rectangle that has the area of "16 cm^2" on a grid paper." As was described earlier, Yuka had difficulty in drawing square and rectangle of the area of "16 cm^2," and finally relied on counting the number of grids. Similar to Yuka, Owa started to draw a square of 8 by 8. But through conversation with her neighbor, she soon recognized it was wrong. Moreover, she knew that not only square but also rectangle satisfies the condition, and that more than one rectangle can do it (see protocol below).

Owa: (turning to her neighbor) have you done?
Neighbor: Is it 8 and 2 ? ... because, [you should find] multiplication which results in 16...
Owa: But you should find the square... oh! (erasing the rectangle (8 by 8), and drew a rectangle (8 by 2) and a square (4 by 4)) ....
Neighbor: Let's see... how about the rectangle lying down (meaning a rectangle of 2 by 8)?
Owa: But [you should find] the rectangle... what?... (listening to someone) can you fine 4 such rectangles?

Especially, once Owa knew that there would be more than one rectangle, she began to work hard to find all of them. She asked her neighbor, "How many of them have you found?"

and "Do you know the tall one and the short one is the same?" and also mumbled "…, well, well…(she can't find the last one)." Here, Owa tried different combinations of two numbers whose multiplication resulted in 16. She used multiplication as the primary tool of thinking in order to achieve her goal of finding all the combinations. In contrast to Owa, Yuka did not share the goal of measuring the area by utilizing multiplication.

Behaviors of Kenji in approaching the tasks in class. As described above, Kenji stayed at the "label" use. When following the traces of his behaviors closely, it is notable that he rarely shared the goal of measuring area by making use of multiplication in the class. Indeed, there was only one occasion in which he shared the goal for a T-shaped figure (see Figure 4). He was observed to look at the figure seriously and began to develop a mathematical expression up to "6x3+2x2+." Actually, he was very close to the goal by way of "decomposition." However, all the obstruction by his neighbor let him quit his thinking. Moreover, later in the class he erased his work and copied the one written on the blackboard ("removal" way) by the teacher. After all, he failed to achieve the goal on his own. After this, he had never tried to share the goal and participated only in the part of calculating the answer. To measure area was degenerated into multiplying two numbers in any way.

Discussion

Relating to internalization, Wertsch and Stone say, "One of the mechanisms that makes possible the cognitive development and general acculturation of the child is the process of coming to recognize the significance of the external sign forms that he or she has already been using in social interaction" (1985, p.166). This was seen in the pupils' use of multiplication in the mathematics classroom. The three phases described in this article stress that it is a constructive process that involves several important generations and modifications of sense and significance toward "multiplication for measuring areas."

The first phase of use of multiplication suggest that the pupils saw and used it without relying on it in their thinking. This phase would correspond to the phenomenon that the tool is located outside the pupil (Cobb, 1995, p. 381). For some pupils, like Yuka, this phase was also important to proceed to the next phase, because it enabled them to develop a sense and significance, which they could modify later. In the second phase, the pupils came to recognize that the multiplication has a legitimate meaning in its own right. Here its use was generalized into compounded figures and different units of measure. Such generalization was supported by the referent of the notation "axb" developed by the pupil. In the case of Yuka, it was the rectangle (a by b). In the third phase, their concern was not to use multiplication any more, but to measure areas. Here their use of multiplication became flexible and purposeful. The second and the third phases suggest that in the process of the internalization, awareness was constructed in two levels. One was the awareness of multiplication as "the" way of measuring areas. This was seen in the second phase. The other was the awareness of multiplication as "a" way of measuring areas, and it was seen in the third phase. In the latter, there was a reflective shift of attention from multiplication in action to multiplication as object of thinking.

These phases were seen in the midst of the pupils' interactions with tasks, peers, and the teacher. Comparisons of some of the pupils' use of multiplication revealed that the goals the pupils tried to achieve in using multiplication differed largely. Based on them, it may be conjectured that when the pupil successfully accomplishes the goal on behalf of multiplication, certain awareness takes place and thereby, the transition to the next phase becomes possible. The pupils would make access to multiplication by all the help given by their peers and the teacher, but it must be themselves who reduce the shared process and product to their own thinking (c.f., Waschesocio, 1998). The result of this study also shows that the interpretation-activity of classroom task, which is often ambiguous, need to be scrutinized further to clarify the condition for establishing the desirable goal with respect to the specific tool use.

Concluding Remarks

In this article, the process in which the pupils acquire new use of multiplication was analyzed and characterized by three phases. The phases show the pupils' meaning construction of multiplication in the context of measuring areas from a secondary accompaniment to a powerful tool of thinking. Individual differences among the pupils in terms of the phases were further considered, and large difference in their task-interpretation activity became apparent.

One contribution of this finding to instructional practice is that pupils' use of new tool should be taken seriously in the classroom. At one time, we may be indifferent to pupils' tool use and leave some to the first phase throughout the instruction. At other time, we may push pupils to the third phase almost just after the introduction. The three phases show that the tool use evolves during instruction, and it is through different experiences of reflecting on one's previous sense and use. We also tend to discourage pupils peculiar tool use, as seen in the first phase, as wrong. However, such use also plays a role in enabling the next phase happen. We need to externalize pupils' idiosyncratic tool use and provide them opportunities to talk about them.

The characterization proposed in this paper is based on a very small number of students. Therefore, further information must be collected in order to nurture the hypotheses. In this regard, one interest is to investigate the condition of the second phase when the transition to the third phase takes place. In the case of Yuka, she engaged in successful experiences of using multiplication by herself in different problems, together with talking and writing about her tool use. Investigation of communication process in light of the phases will give further information. The notion of tool needs to be explored as well. Especially, the goal a person is trying to achieve with respect to the tool, systems of signs used to denote the tool, and syntax and semantics of them (Hino, 1998) subtly influences the person's tool use. These explorations will reveal the details of the relationship between the person's tool use and his/her mathematical development.

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UNDERSTANDING LINEAR ALGEBRAIC EQUATIONS VIA SUPER–CALCULATOR REPRESENTATIONS

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For many students, understanding of linear algebraic equations primarily accrues from working in a single, symbolic representation. It is not until later that they study graphs and attempts are made to link these to the previously studied equations. This paper reports results of a study where super–calculators were used with 13 year–old Korean students, none of whom had used calculators in mathematics lessons before, in an attempt to make explicit links between symbolic, tabular and graphical representations of equations. The results show that the students did build some improved deep links between the domains and that the experience of using the calculators was generally a positive one for them.

Background

It appears that often the focus of mathematics for secondary students has been on learning which encourages them to build knowledge specific to particular problems. Further, much of this learning has emphasised procedural methods rather than concepts. For example, in early algebra sometimes students learn how to simplify, expand, factorise and solve, without understanding the meaning of the processes or the nature of the objects (Tall *et al.*, 1999) upon which they are acting. While procedural knowledge is important it is prone to be learned instrumentally (Skemp, 1979).

There is wide agreement that our conceptual structures or schemas are a key determinant of our progress in mathematics. It seems clear then, that the richer one's construction of schemas in a given domain in terms of conceptual associations or C–links (Skemp, 1979), the greater the potential for future expansion and linking to other domains of mathematical knowledge. One way to accomplish this is by making explicit conceptual links between different representations of mathematics, described by Kaput (1987, p. 23) as involving "a correspondence between some aspects of the represented world and some aspects of the representing world." Kaput (1992, p. 524) has listed one of the four classes of mathematical activity in school as "translations between notation systems, including the coordination of action across notation systems." and explains the value of technology in enabling manipulation of mathematical concepts both within and between these different representations. However, more is required than the ability to translate between the representation systems. Students may have such a surface ability without an understanding of the deeper, conceptual links which are imbedded in the transformation between the representations (Greer & Harel, 1998). Chinnappan and Thomas (1999) have investigated the schemas of experienced teachers who employ modelling techniques, and have suggested a model with schematic conceptual links as the foundation for learning which relates representations, internally and externally.

This research investigated the value of making explicit links between three different representations of linear algebraic equations: the symbolic; the graphical; and the tabular forms. While these representations have often been used before, here the concept of equation, and in particular the deep idea of conservation of a solution to an equation under cross–representation transformations, was emphasised during the learning experience.

A consideration of the best way to approach this led to the graphic calculator, since all three dynamically related representations arise naturally in that context (Kaput, 1992). In addition, graphic calculators are more accessible to students than computers are in many schools (Kissane, 1995; Thomas, 1996) and this is a key advantage. However some have been sceptical of the value of the technology in secondary mathematics learning and continued research is needed in order to provide convincing evidence of how graphic calculators can be valuably employed in the mathematics curriculum. There is already a growing body of such research (e.g. Ruthven, 1990; Penglase & Arnold, 1996; Graham & Thomas, 1997) supporting their use in mathematics learning, but it is still restricted in terms of content area and use of the calculators' facilities.

Method

The research described here forms the Korean part of a study of students in New Zealand and Korea (a high performing country on the TIMSS results), whose aim is to investigate the use of super/graphic calculators to improve students' conceptual understanding of linear algebraic equations. This research was carried out during the period 5th–16th July, 1999.

Subjects: The study involved one class of 35 Form 4 students (aged 13–14 years) from a middle school in Seoul, Korea, who are not currently allowed to use any calculator in their classes, or for assessment, such as examinations. Thus while 27 of the subjects had used a calculator (but never in their standard mathematics lessons) none had ever used a graphic or a super–calculator. The students had previously covered simplification of algebraic expressions and solving linear equations during the school year. This enabled us to use them as a stand–alone single subject group to see what, if anything, they could gain by additional exposure to and linking with the alternative methods of approaching equations that the super–calculator permits.

Instruments: A module of work using the TI–92 super–calculator was prepared. This contained a description of the basic facilities of the TI-92 and then showed how, using a 'Press', 'See', and 'Explanation' format, linear equations can be solved in three different ways: algebraically, graphically, and numerically from a table of values. Two algebraic methods were given, using the TI–92 to solve the equation directly and also using a standard balancing algorithm. An illustrative section from the module showing the four methods for the equation $2x - 5 = 3x - 9$ are given in Figure 1 (note the section is incomplete and formatting has been changed). The fact that the solution is the same in each case was emphasised. Two parallel tests, divided into sections A and B and comprising different numerical values, were constructed as pre– and post–tests. Section A of these tests comprised standard textbooks questions such as: $5x - 8 = 3x + 2$; $m = 8 - 3m$; and $6 - 8n = -3 + n$.

In contrast, section B addressed the students' conceptual thinking in solving equations, both within and across different representations. The concept of equivalent equations is also important and we wanted to know whether the students were able to conserve equation under addition of constant or variable quantities as used in method 1b), and could recognise equivalent ones without having to find solutions.

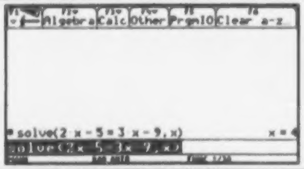
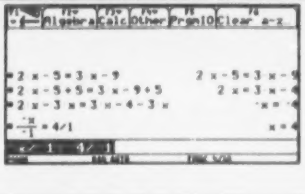
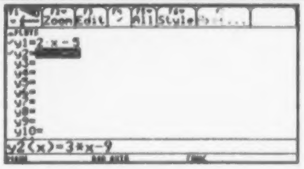
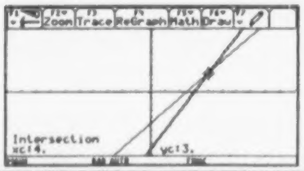
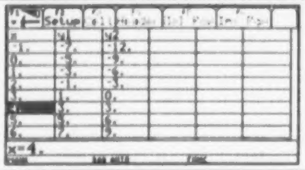
Method 1 a) [◆] [HOME] F2 1 2x [−] 5 [=] 3x [−] 9 [,] x [)] [ENTER]	 solve(2 x - 5 = 3 x - 9, x) x = 4 solve(2x-5=3x-9,x)	$2x - 5 = 3x - 9$ is solved by an algebraic method. The [,] x tells the calculator to solve with respect to x. $x = 4$ is the value which makes both sides equal in value.
Method 1 b) 2x [−] 5 [=] 3x [−] 9 [ENTER] 2x [−] 5 [+] 5 [=] 3x [−] 9 [+] 5 [ENTER] 2x [−] 3x [=] 3x [−] 4 [−] 3x [ENTER] [(-)] x [÷] [(-)] 1 [=] [(-)] 4 [÷] [(-)] 1 [ENTER]	 2 x - 5 = 3 x - 9 2 x - 5 = 3 x - 9 2 x - 5 + 5 = 3 x - 9 + 5 2 x = 3 x - 4 2 x - 3 x = 3 x - 4 - 3 x -x = -4 $\frac{-x}{-1}$ = 4/-1 x = 4 -x/-1=-4/-1	To find the value of x, we need to simplify the given expression step by step: If we add 5 to both sides, the expression is simplified to $2x=3x-4$. If we subtract $3x$, the expression is simplified to $-x = -4$. If we divide by −1, finally we get $x=4$
Method 2 [◆] [Y=] 2x [−] 5 [ENTER] 3x [−] 9 [ENTER]	 PLOTS y1=2 x - 5 y2= y3= y4= y5= y6= y7= y8= y9= y10= y2(x)=3*x-9	Here each side of the equation is defined as a function, using $y1(x)$ and $y2(x)$: $y1(x)= 2x - 5$ $y2(x)= 3x - 9$ Looking at the two graphs, we can see that they intersect at one point.
1st curve? [ENTER] 2nd curve? [ENTER] Lower bound? 0 [ENTER] Upper bound? 6 [ENTER]	 Intersection xc:4. yc:3.	1st curve means $y1(x)$, 2nd curve means $y2(x)$. The lower and upper bound means the interval in which the intersection point is found. So the two graphs intersect at the point (4, 3). i.e. $x = 4$
Method 3 [◆] [TblSet] tblStart: −1 Δtbl: 1 [◆] [TABLE]	 x y1 y2 -1. -7. -12. 0. -5. -9. 1. -3. -6. 2. -1. -3. 3. 1. 0. 4. 3. 3. 5. 5. 6. 6. 7. 9. x=4.	The point of intersection can be found using a table. Enter y1 and y2 as in method 2. When we look at the point $x=4$, we can see the values of the two functions are the same, and equal to 3.

Figure 1: A section of the module showing the layout and calculator screens

Within the symbolic representation we asked them questions (B1) such as:

Do the following pairs of two equations have the same solution? Give reasons for your answer.

a) $5x - 1 = 3x + 2$	b) $2 - 3x = x - 3$
$5x - 1 + 8 = 3x + 2 + 8$	$2 - 2x = 2x - 3$

We note that in both examples above the second equation may be seen by someone with the concept of equivalence of equations as a *legitimate transformation* of the first (by adding 8 or x to both sides) conserving the solution, although this may not be a *productive transformation* in terms of actually obtaining that solution.

In addition the tests required the students to maintain the concept of solving an equation across representations, asking them to solve a symbolically presented equation in a graphical and a tabular domain (question B7) and to convert a graphical picture into a symbolic representation, as in Figure 2.

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B6. Write a single equation in x which can be represented by the following diagram:

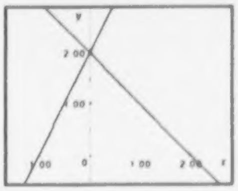
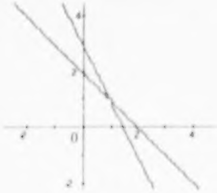


Figure 2: A question assessing understanding of transformation from a graphical to a symbolic representation

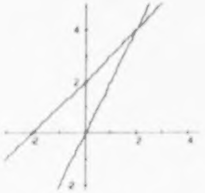
Students were also given a list of 6 symbolic, 3 graphical and 3 tabular representations and asked 'Which of the following are different ways of representing the same equations?' (see Figure 3). This tested their ability to conceptualise functional equality across representations.

B2. Which of the following are different ways of representing the same equations? Match the letters which correspond and write them in the boxes below.

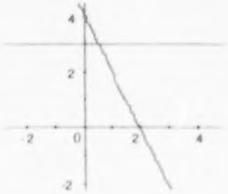
$x + 2 = 2x$	$4 - 2x = 3$	$-2x + 3 = 2 - x$
A	B	C
$5 - x = 2 - 2x$	$2x - 4 = 4x + 10$	$5 - 3x = -4$
D	E	F



G



H



I

x	y
-1	5
0	3
1	1
2	-1
3	-3

x	y
-1	3
0	2
1	1
2	0
3	-1

J

x	y
0	5
1	2
2	-1
3	-4
4	-7

K

x	y
0	4
1	2
2	0
3	-2
4	-4

x	y
0	3
1	3
2	3
3	3
4	3

L

Answer ☐ ☐ ☐ correspond ☐ ☐ ☐ correspond

☐ ☐ ☐ correspond ☐ ☐ ☐ correspond

Figure 3. A question testing equivalent representations of equation forms

As well as the test each student was given a questionnaire covering their experience with the calculators (serving to triangulate the test results) and their understanding of the concepts associated with linear equations, and an attitude scale (using a 5 point Likert format) on their feelings towards calculators.

Procedure: The module (written in English and translated into Korean – the English version is shown in this paper) was initially given to the class teacher, who familiarised herself with the content. The first named researcher met twice with her to answer her questions and to make sure that she was comfortable with the calculator and the material. The teacher then taught the class for four lessons, two covering basic facilities of the calculators including introducing graphs and tables, the other two describing how to solve equations in different ways on the TI–92. The only previous experience the students had had was in solving equations algebraically.

Each student had access to their own TI–92 super–calculator, which they kept with them for the whole of the time of the study, including their time at home. During lessons the teacher stood at the front of the class, who sat in the traditional rows of desks, demonstrating each step while the students followed in the module and copied her working onto their own calculator. She employed a calculator viewscreen and projected the image using an overhead projector. However, the calculator commands were projected in English and it was necessary for the teacher to translate each one into Korean for the students, causing some confusion until they became more accustomed to the English commands. Each session lasted 45 minutes, and after the teacher's explanation, the students spent the rest of the session practising while the teacher circulated and assisted with any problems, and the last five minutes were used for a summary. At the end of fourth tutorial the students were given the post-test, followed by the attitude test and the questionnaire.

Results

The students had spent some time in normal lessons prior to the research learning how to solve equations in a symbolic arena. Hence it was not surprising that they had pre–test facilities of 86.7% and 69% on the two skills questions in section A, showing that before the study they were able to solve even difficult linear equations. However, for question 6 in section B (see Figure 2), not one student could relate either of the lines to an equation in x (this was considered by the teacher to be at too high a level for her students). Even when given a graph or tables and an accompanying symbolic equation as in question B7 (see Figure 4), very few students could solve the equation using the graphical or tabular representation, this question having a facility of 15.8%. On question B2, (Figure 3), students also had difficulty, with just 11.5% of answers correct. It seems, from this pre–test evidence, that students were good at solving equations symbolically, but were less good at translating between the symbolic and other representations, or using those representations to solve equations. They had problems relating the surface features of the domains and had not built an understanding of the deeper relationship based on conservation of equation solution under different representations of the functions.

After the calculator intervention there was no change in the students' skills at solving the equations (Section A: max score = 7; $m_{pre}=5.61$, $m_{post}=5.54$, $t=0.16$, n.s.) but they had improved significantly on some of the more testing questions in section B (max score = 35; $m_{pre}=13.1$, $m_{post}=17.0$, $t=2.25$, $p<.05$). However, the improvement was variable across the questions, as may be seen from Table 1, which gives the mean scores for each of the questions in sections A and B.

Each of questions B2, B6 and B7, where the students performed significantly better after the module, involved relating data between two or more representations (see Figures 2 and 3). While the improvement in question B6 was small, involving only 5 students, this was a more difficult question, needing further attention. Figure 4 shows the response of student L2 on question B7, where they were asked to solve the symbolic equation using the given graph or table. On the pre–test he was unable to use the graph or table to solve the equation, but on the post–test he has made the link to a solution in each case, and understands why it is the solution.

Table 1: A comparison of pre– and post–test section A and B mean scores

Question Number (Max score)	Pre–test mean	Post–test mean	*t*	*p*
A1 (5)	4.33	4.09	–0.44	n.s.
A2 (2)	1.38	1.46	0.41	n.s.
B1 (18)	10.5	11.5	0.77	n.s.
B2 (4)	0.46	1.14	3.95	<.0005
B3 (2)	0.85	1.09	1.46	n.s.
B4 (1)	0.04	0	–1.79	n.s.
B5 (2)	0.35	0.53	1.46	n.s.
B6 (2)	0	0.17	2.24	<.05
B7 (6)	0.95	2.66	3.29	<.005

We note that for the graph question he also solves the equation symbolically, but does not do so for the table.

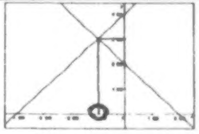
Pre–Test Solution	Post–Test Solution
$x+x=3+1$ $2x=4$ $x=2$ 정답 = $x=2$	 정답 = $x=-1$ 왜냐하면 x축에 –1.00 이라고 표시 되어 있다. [*x* equals –1, because –1.00 is indicated on the *x*-axis]
No Response	정답 = $x=1$ 왜냐하면 두 표가 나머지는 다 다른데 x의 1과 y의 2가 같다. [$x = 1$, because the values of *y* as 2 at the point $x = 1$ are the same from the two tables]

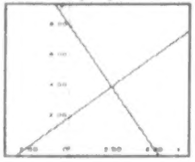
Figure 4: The B7 work of student L2 showing conceptual links between representations

The aspect of conceptual understanding, seeing the relationship between equations and their solutions across the representations, was further investigated in the students' questionnaire, where they were asked the following question:

Is there a relationship between A, B, C in following diagrams? If so, then what is it?

$x + 2 = -2x + 8$

A



B

x	*y*
–1	1
0	2
1	3
2	4
3	5

x	*y*
–1	10
0	8
1	6
2	4
3	2

C

This was very similar to question B2, where the pre–test facility was 11.5%, and the post-test 19%. In the questionnaire 17 (48%) of the students answered that the value of *x* is common, showing that they may have conserved solution of equation across the representations. It could be argued that they had merely calculated that in each case the solution was 2, without appreciating that the functions were the same. However, 9 (25%) of the students also answered that the expression A can be shown by the graph B and the table C, demonstrating that these had made the link.

Further supporting this, 11 (31%) of the students mentioned later in their questionnaires, as an advantage of calculators, that through the use of the table, graph and algebra the solution to the equations could be more easily and quickly understood. In addition, when asked 'How many different ways can an equation be represented?' 21 (60%) replied 3, citing algebraic, table and graph. This was evidence that the tutorial had assisted construction of schematic understanding of surface and deep relationships between algebraic, tabular graphical representations of the concept of equality of linear functions.

Table 2: Breakdown of methods employed on the conservation of equation question B1

Pre-test	No solution*	Solve	Solve+Explain	Explain only
Method	39 (18.6%)	19 (9.0%)	109 (51.9%)	43 (20.5%)
Post-test	No solution*	Solve	Solve+Explain	Explain only
Method	45 (21.4%)	23 (11.0%)	48 (22.9%)	94 (44.8%)

* or incorrect solution

At first sight it appeared (see Table 1) that there was no improvement in the test question (B1) on the conservation of equation under transformations in the symbolic representation. However, as Table 2 shows, in the post–test many more equations were correctly solved by students in question B1 by considering the relationship between the two equations, without needing to solve the equations (as shown by the 'explain only' column). Student C, for example, had to symbolically solve equations of the type in question B1 a) and b) in the pre–test. However, in the post–test she was able to write that they have "the same solution because if you add the same value to both sides then the equation is the same", and "If you multiply both sides by the same value [4] then the equation is the same", without needing to solve either equation.

Student Attitudes: These students had never used a calculator in their mathematics learning, hence it was of great interest to find out their view of them. After their tutorials, when asked 'How do you feel about the TI–92 graphic calculator?', 22 (62%) of the students replied that they felt easy, comfortable, curious or were interested in using the calculator. When asked what difficulties they had encountered 17 (48%) of the Korean students replied that the commands on the calculator were difficult because they were in English. The attitude scale questions confirmed the positive view of calculators. Each question was scored with an integer from 1 to 5, and scores were reversed on negative questions so that in every case the higher the score the more positive the attitude to calculators. Overall their response to the calculators was significantly positive, with a mean score of 3.54 (t=5.16, p<.00005). Their mean scores showed that they clearly thought that calculators were valuable (I think the calculator is a very important tool for learning mathematics, 4.0), and made mathematics more interesting (More interesting mathematics problems can be done when students have access to calculators, 4.2). Further they were keen to use them more themselves (I want to improve my ability to use a calculator, 4.17), and thought that others should learn how to use them (All students should learn to use calculators, 4.0). It was interesting to see that they were not really influenced by commonly held views, such as the detrimental effect of calculators on mathematics skills (Using calculators will cause students to lose basic computational skills, 3.06; Students should not be allowed to use a calculator until they have mastered the idea or method, 3.23).

Conclusion

Our contention is that important conceptual links between the symbolic, graphical and tabular representations of functions can easily be lost if algebra is approached in a purely procedural manner. The value of graphic and super–calculators is that they may be used to assist teachers to make these links explicit, provided teachers are pedagogically alert to the deeper, underlying conceptual relationships, and preservation of the conceptual structure of the mathematics is central to their schemas. For example, one may approach the graphical solution of linear equations by providing the surface, procedural method of drawing the graph of each function and reading off the x–value of the point of intersection without explicitly tackling the deeper, functional relationships (see Greer & Harel, 1998 for other examples). The concept of linear function passes across four different representations in this method: algebraic, tabular, ordered pairs and graphical. To build rich relational schemas which contain internal representations of the external ones, students should experience the links and the sub–concepts of one–to–one, independent and dependent variable, etc. in each representation. In addition, the fact that two one–to–one functions coincide for a single value of the independent variable, and the conservation of these values across any representation is important. In this research study we have begun the process of testing the value of super–calculators such as the TI–92 in promoting the construction of relationships like these. This small–scale, uncontrolled study, provides some evidence of the value of the approach, with the students able to add conceptual schematic links to the knowledge they had built by studying equations purely symbolically. Whether the calculator is a significant factor, or the results could be duplicated without using them is of considerable interest, and we would welcome a study which sought to determine this.

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THE RELATION OF MOZAMBICAN SECONDARY SCHOOL TEACHERS TO A MATHEMATICAL CONCEPT: THE CASE OF LIMITS OF FUNCTIONS

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The teaching and learning of the limit concept has been the object of study for various Mathematics Education researchers. Some of these researches focused on the issue of students' conceptions (Tall and Vinner, 1981; Cornu, 1981; Sierpinska, 1981), others on the epistemological issue (Cornu, 1982, 1983, 1984; Sierpinska, 1985) and still others on the results of didactical engineering (Robinet, 1983; Trouche, 1996).

In this paper we present some results of a research concerning Mozambican teachers' conceptions about limits of functions and its teaching. This study constitutes the second phase of a four-year research project, which aims to improve the teaching and learning of limits of functions in the Mozambican secondary schools.

THEORETICAL FRAMEWORK

As a main approach for the research project as a whole, we used the notions of *didactical anthropology* (Chevallard, 1992) and of *interplay between settings* (Douady, 1986).

Chevallard' s theory of didactical anthropology pointed out the relation between knowledge and institutions, the word "institution" being used in the broad sense of the term: it can be a school, a class, but also "practical work", "lectures", "family" and others (Chevallard, 1992, p. 144).

> *The new presentation of the theory is based on the central theme of knowledge and institutions, all knowledge being knowledge from an institution, the same knowledge being able to "inhabit" different institutions, on condition that it adapts to these institutions, hence the notion of ecology of knowledge: for a knowledge to live in an institution, it has to submit to a certain number of constraints which basically means that it must alter or it cannot remain in the institution (Arsac, 1992, p. 123).*

The "institutional relation" to an object of knowledge influences the establishing of individual relations to this object. Therefore, an individual establishes a personal relation to a certain object of knowledge if he/she has been in contact with one or several institutions where this object of knowledge is found. A Mozambican Mathematics teacher, for exemple, has been in contact with the limit concept through different institutions, as a secondary school pupil some years ago, as a university student, as a secondary school teacher, etc. Therefore, his/her personal relation to limits of functions depends on various institutional relations to this concept. He/she is supposed to teach in accordance with the relation that the Mozambican Secondary School has to this object but his/her individual relation may fit or be in contradiction with it.

Douady showed that a mathematical concept could be studied in different settings, e.g. algebraic, numerical, graphical and geometrical settings. Each setting allows to associate different mental images with this object and to develop different problematics. *The interplay between settings* consists of shifts of settings induced by the teacher in order to help the pupils' conceptions to evolve.

> *The shift of settings is a way to obtain different formulations for the same problem. These formulations are not necessarily exactly equivalent but allow a new access to the difficulties and the use of tools and techniques which were not obvious in the first formulation* (Douady, 1986, p. 11, our translation from French).

For instance, the limit concept can be studied in a very formal way, using the ε-δ definition, but can also be studied in a very intuitive way. In the numerical setting, the students can use numerical values in order to approach a limit and understand what it means that the limit of a function has a finite or infinite value. They can also use a graphical setting in order to understand what a given limit means for the graph. The limit concept can also be handled in an algebraic setting, calculating limits by using algebraic techniques. All these different aspects of limits of functions are complementary and shifts of settings would help the students to reach a deeper understanding of this concept.

Our study of the Mozambican secondary school system's relation to the limit concept seems to reveal that it is mainly considered as an object of study whose theoretical approach is divorced from what is expected in practice from the students. It seems that the limits of a function are not considered very useful for sketching its graph or vice-versa. What is basically required of the students is to handle limits of functions in an algebraic setting (Huillet, Mutemba, 1999).

In this paper we will present some results of a study aiming to detect regularities and differences between the relations of Secondary School teachers to limits of functions, specifically in relation to different settings.

METHODOLOGY

Subjects

This study involves all nine Mathematics teachers who are teaching Mathematics in grade 12 in public schools of Maputo and Quelimane. 8 of them are professional teachers, 2 with a "Bacharelato" (three-year university course) for teaching Mathematics and Physics, and 6 with a "Licenciatura" (five-year university course). The 9th teacher, from Quelimane, is an .

In the first phase of the study these teachers were asked to answer a questionnaire. The teachers from Maputo were interviewed later in order to explain and deepen some of their answers and answer other questions that emerged from the analysis of the questionnaire. For various reasons, it was not possible to interview the teachers from Quelimane.

Instruments

Questionnaire

A questionnaire was developed using the results of the study of the Mozambican Secondary School institutional relation to the limit concept (Huillet, Mutemba, 1999) and focusing on the different settings that can be used in the teaching of limits of functions. The questions aimed to detect:

- how the subjects actually use the different settings (formal, numerical, algebraic and graphical settings) in teaching this item;
- what they expect that students will be able to do in relation to this concept;
- which difficulties they face in teaching this concept;
- which difficulties they think that the students face in learning this concept;

- what they think about the current practice of teaching this concept at Mozambican secondary schools.

In an a-priori analysis we tried to formulate our expectations about the teachers' answers, based on the previous analysis of the syllabus, textbooks and national exams and on our experience as teachers and/or teachers' trainers. The questionnaire was tested with a grade 12 teacher of a private school in Maputo. After some modifications it was administrated in the two towns in August 1998.

The teachers' answers to the questionnaire were analysed individually by each of us and afterwards our analyses were discussed and deepened.

Interviews

The interviews, which aimed to deepen the teachers' point of view about the teaching and learning of limits of functions in Grade 12, were structured interviews. For each interview, we used:

- a general guide based on the teachers' answers to the questionnaire and indicating general questions to be focused;
- a specific guide based on the individual's answers to the questionnaire and indicating specific questions.

The interviews were conducted from March to June 1999. They were audiotaped and transcribed. At this moment the transcripts are being analysed. In this paper we will use only those results of the interviews that were available at the time of writing.

FINDINGS

We will present the results of the analysis of the questionnaire and of some interviews according to the different settings we previously identified as relevant for the teaching and learning of limits of functions: formal, numerical, algebraic and graphical settings.

Formal setting

Eight of the nine teachers assert that they teach the ε-δ definition. They do so in accordance with the syllabus, which recommends that the definition be taught although it is "*very abstract and impossible to understand for the pupils*" (Ministério da Educação, 1993, p. 42). Five of them want the students to know the definition but only one says that the students understand it.

During the interviews we asked the teachers whether they think it was necessary to teach the definition at secondary school since they are aware that it is too difficult for the students. One of them, Alberta, claimed that it was necessary for what could be called "mathematical reasons":

> *I think that it is necessary. (...) You must have a definition! You must set out from a basis! (...) Because if you start at one point, you don't have a definition, from the start to write, what are you going to write? You can't write anything, you have to study the theory!*

It seems that this teacher has a conception of Mathematics as a well-organised body of knowledge, where definitions come first, and that the teaching of Mathematics has to follow this structure.

Numerical setting

In order to find out whether the teachers work with the students in the numerical setting, we introduced in the questionnaire the question "*Do you sometimes use calculation with numerical values? Give some examples of the way you use it*". With this question we meant choosing a sequence of values for x approaching a and verifying that the corresponding values of $f(x)$ approach b. We were expecting that at least they would use this procedure to establish the special limit $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$, as suggested in the syllabus. The analysis of the teachers' answers seemed to reveal that they didn't understand the question. Some of them gave examples such as $\lim\limits_{x\to2}(2x+1)=5$, $\lim\limits_{x\to2}\frac{2x^2+x+1}{x}=\frac{11}{2}$, where it is just necessary to substitute the value of x, or said they only use these procedures with sequences. Nobody mentioned the special case suggested in the syllabus. After explaining what we had in mind with this question, during the interviews, most teachers said they don't use numerical values. Some of them use rather complicated ways of getting the specific limit $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$; others just give the results in order for the students to apply it.

Algebraic setting

All the teachers claimed they give much importance to the training of algebraic techniques for indeterminate forms. The most important techniques appointed by the subjects are exactly the ones that we identified as being asked more often in the national final exams (Huillet, Mutemba, 1999). Only one of the teachers claimed that he teaches L'Hôpital's Rule, which reduces the number of techniques. All of them said that the students have difficulties in applying these techniques. Most of them are algebraic difficulties: factorization, cancellation, rationalisation, calculation with powers, roots, etc. Three teachers said that the students mix up the different techniques used to solve indeterminate forms such as $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

During the interviews we asked the teachers why they think these techniques are so important. Our argument was that, after entering university, the students would not need them anymore because they would learn a more powerful tool: L'Hôpital's Rule. The answers of some teachers seem to reveal that they see all these techniques as an application of the algebraic ones as we can see by the following extract of Alberta's interview:

> *A: (...) I don't know whether it would be worthwhile to simplify like that, only by L'Hôpital's Rule without seeing other things.*
> *I: I don't know. What do you think?*
> *A: Because, in the end, ... all these limits, for example they* [the students] *are not going to factorize anymore!*
> *(...)*
> *A: It is because in grade 11 we teach how to factorize. In grade 10 they only factorize the quadratic form, but in grade 11, third ... forth, fifth degree. (...) So such factorization we use it here!*

It seems that, for this teacher, things have been inverted: instead of teaching the pupils how to factorize in order to use it in other mathematical activities, factorization became the principal objective, the others mathematical activities, such as calculating limits of functions, becoming an application of factorization.

Graphical setting

The analysis of the teachers' answers to the questionnaire shows that they only use the graph when introducing limits of sequences and in some tasks that require reading the limits of a function from its graph. Usually the limits of a function are not used to sketch the graph. Seven teachers said that they introduce the asymptotes before the limit concept, in accordance with the syllabus, when studying rational functions such as $f(x)=\dfrac{ax+b}{cx+d}$. Only one subject mentioned exponential and logarithmic functions. Explanations given by the teachers about the link between limits and asymptotes seem to reveal that they consider the determination of asymptotes as a mechanical process. For instance Alberta wrote in the questionnaire:

> *A.V.* [vertical asymptote] → *related to the domain of the function*
> *A.H.* [horizontal asymptote] → *related to the range*

This conclusion is consistent with observations made in classes at university, where students usually declare that $x = a$ is a vertical asymptote when a does not belong to the domain, without calculating the limit.

During the interviews, some teachers noticed that the students are not able to link the result of a limit to the graph of the function because they are not used to do it. Eugénio, for example, declared that the teachers are responsible for that, because they do not work in that way.

DISCUSSION

The personal relation of the teachers who filled out the questionnaire to the limit concept seems to fit with the institutional relation of the Mozambican Secondary School as studied through the syllabus, the textbook and the tasks about limits in the national exams. This relation has mainly two components:

- a formal one, derived from mathematical reasons which induces to teach the formal definition of limits even being aware that the students do not understand it and are not able to apply it;
- an algebraic one where the students are asked to calculate indeterminate forms of limits, mainly because of the national exam's tasks, as stated by Filomena: "*because at the exam it is only calculations, yes it is only calculations, so we teach more calculations*".

The numerical and graphical settings are hardly used in the teaching and learning of the limit concept. Some of the teachers do not question the way they teach limits, arguing that they prepare the students for the national exam. Others seemed to reflect upon this situation during the interview. For instance Eugénio, when asked, at the end of the interview, what kind of tasks he would suggest for the national exams, answered:

> *Well, I would suggest limits where there is a lot of application in terms of sketching graphs*

This answer came into conflict with his earlier statement that he does not use many graphs when teaching this concept.

However we found some differences between the individual teachers' relations to the teaching of limits of functions. Some of them asserted that they only teach the techniques, without any explanation or demonstration. That is e.g. the case of Filomena, who stated *"Here we only give the formulas"*. Others try to demonstrate them. For instance Eugenio and Gustavo each have their own demonstration for the special limit $\lim_{x \to +\infty}\left(1+\frac{1}{x}\right)^x = e$, which they explained during the interviews.

At this moment we still are analysing the interviews' transcripts. A deeper analysis might provide more information about differences between the teachers' relations to limits of functions.

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THE MEANING OF TERMS CONCERNING THE TIME ORDERING FOR FIRST GRADE STUDENTS: THE INFLUENCE OF CULTURAL BACKGROUND

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Abstract

This paper presents a research that intended to evaluate the knowledge of students of the 5th year of the first grade (10 to 11 year-old) about ordering relations in time and the possibility of evolution of their knowledge. The research was applied in two groups of students, totaling 66 students, from schools of the state of São Paulo, Brazil. The students had a pre-test, a class and a post-test based on the Didactic Situations Theory. This study indicates that the students could evolve from a restrict conception of order relations in time (cultural usage of terms such as arriving before somebody) to a mathematical (non-restricting) meaning of this order relation.

1. Introduction

The relationships with time, along the cognitive development of the human being, have been the object of study of many Cognitive Psychology researchers. We can mention the works by the Geneva group, which aim, mainly, at defining and explaining the conservation of duration through the Developmental (Phases) Theory (Piaget, 1966). Vergnaud and Errecalde (1980) also researched the representation of dates in straight lines, with 9-13-year-old students. They concluded that none of the recordings obtained would be close to the correct one, since the objects represented required from the students a synthesis of concepts and properties of order, distance and interval. Maranhão (1996), as opposed to the Piagetian theory mentioned above, considers that it is necessary to artificially establish (through the durations along line segments), a convention which enables a decision criterion about the equivalence of both durations, in order to compare them and thus, talk about their conservation. Agreeing with Vergnaud, she builds up a synthesis among order, distance and interval, considering the problems concerning time as derived from a didactic developmental process rather than from a purely psychological one. In the above mentioned research, she investigates various conceptions of students (aged 9 to 12) about time, and particularly, ordering. She detected some difficulties of the students, such as: mixing up *arriving early with arriving late;* considering that the term *late* means *less (minus)* in the calculation of time; considering that *early* means *ahead* (in the representation on an oriented axis), or *late* means *behind* (in the representation on an oriented axis). One of the interpretations of the researcher about the confusion the students made was that the terms *early* (adiantado, in Portuguese) and *ahead* (adiante, in Portuguese) have the same lexical root, which could be associated with *ahead* and therefore, *more* in terms of hours. *Delay*, in its original root in Portuguese (atraso) may be associated to *behind* (atrás) and therefore to *less hours*.

In this study we intend to deepen our knowledge about the conceptions of the students about the ordering relation *arriving before somebody*. We chose problems without any numerical references because we took Maranhão's first research (1996) as an assumption that the use of terms concerning time may not be evident in problems which bring numerical references and the meanings attributed to them, by the students, may be a source of error in these problems.

We wanted to know whether there is an influence of the cultural usage of the students' conceptions, in order to find out if they can attribute a broad meaning to terms such as *arriving before somebody*, or on the contrary, if they only attribute a limited meaning to it, using the term *arriving before somebody* as equivalent to *arriving immediately before somebody* to solve a problem which involves ordering relations, without numerical references; whether it is plausible for the students to arrive at the conclusion that *arriving at the same time as* is establishing a comparison in the ordering relation *arriving before somebody*; whether the students attribute a broad meaning to the opposite of these terms (*not arriving after somebody* means *arriving before somebody* or *arriving at the same time as).*

2. Theoretical Framework

The theoretical framework for the research under discussion was developed according to the principles of Brousseau's Didactic Situations Theory (1997). In the typology of the situations, the action situations are understood as researching ones, aiming at the knowledge of a mathematical object; the formulation situations are regarded as ones of explanation of concepts by the students, usually produced because an action situation so requires. The validation situations are understood as a confrontation of conceptions explained by the students, either through debates among themselves or questioning by teachers/researchers. A problem regarded as a source for learning must lead the student to a reflection, which involves him in the action dialectics. It must be conceived so that the student has the knowledge to solve it, at least in part, and that some mathematical knowledge is crucial for the complete solution; the teachers/researchers, as important elements in the didactic situation, also create classroom conditions to activate the dialectics of formulation and validation, the latter two contributing for the cognitive evolution. This author considers the evolution of mathematical knowledge as a cognitive one.

Therefore, the problems proposed to the students in this study were conceived in such a way as to benefit from their cultural background. Therefore, they were derived from their daily routine.

In their descriptions, there were 4 characters to be ordered. If there were only two characters, *arriving before somebody* would be equivalent to *arriving immediately before somebody.* The proposed description enabled, in the validation situation, the understanding of how to order 3 characters. The inclusion of a 4th element enabled the focus on the meanings attributed to the terms, leading to the consideration that there were several correct answers for its positioning, if we used the mathematical (non-restricting) meaning of these order relations.

3. Methodology

We conceived and applied a test, with problems, about an ordering in time, involving four characters. The test was applied in two groups of students of the 5th year of the first grade (10 to 11 year-old), totaling 66 students, from schools of the state of São Paulo, Brazil. There were three application phases, spaced in time one week from each other, always conducted by the same researcher and based in the theoretical framework. In the 1st phase, the students solved the problems with paper and pencil, individually. In the 2nd phase, we promoted discussions with groups of students of both classes (in each class all students were present), from the answer sheet of each student, filled out in the previous phase. They received the answer sheets filled out in the 1st phase and used a blue pencil to answer the questions again, now according to the group. Besides that, we promoted a debate (general discussion with the whole group) about the outcomes of the students. The discussions were held in two steps, for each problem. In the first step, we discussed effective strategies for ordering, such as the use of an arrow corresponding to the ordering relation *arriving before somebody* (for instance, if the group decided that *before somebody* should correspond to the positioning *on the left,* an arrow was drawn, pointing to the left and on it was written *before somebody*). Besides that, any positioning was checked, for each character ordered, against the text of the problem. In the second step, the teacher/researcher conducted discussions with the students, aiming at overcoming the use of the terms in a limited sense. In the 3rd phase, the students solved the problems, whose descriptions had been altered only by changing the name of the characters. We obtained data from the answer sheets filled out by the students and through the notes of an observer, who was present at the debates.

The data obtained from the tests of phases 1 and 3, were coded and analyzed considering their knowledge of ordering, the broad meaning of the term *arriving before somebody* and assuming the fact that two characters arrive at the same time in a sequence.

3.1 The Test

A teacher wanted to know the sequence in which some of her students arrived. They gave her some information and **she could not determine their exact arrival ordering.** Try to find out the possible arrival orderings, according to the information these students provided the teacher.

1. Maria said she arrived at school before Eni. Eni said she arrived at school before Bia. Rita said she did not remember about the other colleagues, but she was sure she arrived at school after Eni.

a) With this description, can we write down an order in which they might have arrived? If this is possible, write it down.

b) According to this description, can you conclude that there is only one possibility for their arrival ordering? If not, indicate one or more possible arrival orderings.

2. Ariel said she arrived at school before Maria. Maria said she arrived at school before Gisela. Rita said she did not remember about the other colleagues, but she was sure she did not arrive at school after Gisela.

a) With this description, can we write down an order in which they might have arrived? If this is possible, write it down.

b) According to this description, can you conclude that there is only one possibility for their arrival ordering? If not, indicate one or more possible arrival orderings.

4. Results

4.1 Descriptive data

4.1.1 Table 1 presents the results assuming that two characters can arrive at the same time in an ordering relation.

Table 1

Assumption that two characters arrive at the same time in an ordering relation, by students of the 5th year of the first grade - 1999 - São Paulo - Brazil

Codes*	Problem 1		Problem 2	
	Phase 1	Phase 3	Phase 1	Phase 3
0	95.45%	36.36%	96.97%	45.45%
1	4.55%	63.64%	3.03%	54.55%

*0 - Does not use *arriving at the same time as* in an ordering; 1 - Uses *arriving at the same time as* in an ordering

4.1.2 Table 2 presents the results about ordering in time.

Table 2

Knowledge of ordering in time, involving four characters, by students of the 5th year of the first grade – 1999 – São Paulo – Brazil

Codes*	Problem 1		Problem 2	
	Phase 1	Phase 3	Phase 1	Phase 3
0	7.58%	3.03%	1.52%	0%
1	9.09%	0%	6.06%	0%
2	19.70%	19.70%	24.24%	12.12%
3	25.76%	9.09%	36.36%	21.21%
4	37.88%	68.18%	31.82%	66.67%

* 0 – makes a mistake in the 3 elements of the two first phrases in both questions; 1 - Makes a mistake in the 3 elements of the two first phrases in only one question; 2 - Is correct in 3 elements of both questions; 3 - Is correct in 4 elements of one question; 4 - Is correct in 4 elements of both questions.

Table 3 presents the results in the knowledge about the meaning of the term *arriving before somebody*.

Table 3

Knowledge of the broad meaning of the term *arriving before somebody*, by students of the 5th year of the first grade – 1999 – São Paulo - Brazil

	Problem 1		Problem 2	
Codes*	Phase 1	Phase 3	Phase 1	Phase 3
0	36.36%	22.73%	31.82%	12.12%
1	13.64%	4.55%	13.64%	15.15%
2	12.12%	4.55%	22.73%	6.06%
3	37.88%	68.18%	31.82%	66.67%

* 0 - coded as 0 , 1 or 2 in the ordering; 1 - A *limited* conception present in one question but not confirmed in the other one; 2 -A *limited* conception in one question and confirmed in the other one; 3 - A *broad* conception in one question and confirmed in the other one.

4.2 Accuracy

The statistical tests were applied, considering the evolution of each student, from phase 1 to phase 3, in each problem and in each question (a) or (b) for each kind of analysis. The results were very significant in all cases. In problem 1, concerning the ordering in time, we obtained by a t-test: $t_c = -3.960773$; df = 65; $p < 0.005$. Concerning the meaning of the term *arriving before somebody* we obtained: $t_c = -3.8495$; df = 65; $p < 0.005$. As regards to *arriving at the same time as* (order) we obtained by the Mc Nemar' s test: $z_c = 6,244998$; $p < 0.001$. In problem 2, as regards to the ordering in time, we obtained: $t_c = -4.681361$; df = 65; $p < 0.005$. Concerning the meaning of the term *arriving before somebody* we obtained: $t_c = -4.22509$; df = 65; $p < 0.005$. As regards to *arriving at the same time as* (order) we obtained: $z_c = 5.830952$; $p < 0.001$.

5. Conclusion

We can see the improvement of student's knowledge in tables 1, 2 and 3, as well as from the accuracy results.

We think that it is important to emphasize that the number of students with codes 0 or 1 (Table 2), who did not order 3 characters correctly in at least one question decreased from 16.67% to 3.03% (in the 1st problem) and from 7.58% to 0% (in the 2nd problem). We understand that the students with these codes lacked understanding about how to order 3 characters. So, we can say that they improved their knowledge.

We interpreted that the students with codes 0, 1 or 2 (Table 2) had some difficulty in ordering or some difficulty to deal with the 4th character. As a consequence of these results, the study of the meaning of the terms was restricted to the cases which present codes 3 or 4 in the ordering. The student's difficulties decreased from 37.28% to 22.73% (in the 1st problem) and from 31.82% to 12.12% (in the 2nd problem).

We observe that the students with code 0 (Table 1) did not assume that two characters arrive at the same time in an ordering relation. We think that they had conceptions influenced by cultural usage, which we expected them to have. We can see great evolution in this cognitive aspect.

We notice that the students with codes 1 and 2 (Table 3) did not present a broad conception of the terms such as *arriving before somebody*. We can say that the students' conception's evolved.

We can finally say that there was an influence of the cultural usage of the students' conceptions and also that they could attribute a broad meaning to terms such as *arriving before somebody. Besides that*, it is plausible for the students to accept that *arriving at the same time as* is establishing a comparison with the ordering relation *arriving before somebody*. We can also say that students attribute a broad meaning to the opposite of these terms (*not arriving after somebody* means *arriving before somebody* or *arriving at the same time as)*. We attribute the students' evolution to the classes conducted by the researcher/teacher, based on the Didactic Situations Theory. We emphasize that the problems were conceived in the same theoretical framework..

In addition, we must consider that the class based on socially oriented models of constructivism as the Didactic Situations Theory and the problems the students dealt with contributed for an analytical cognitive competency.

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The Relationships between Fluency and Flexibility of Divergent Thinking in Open Ended Mathematics Situation and Overcoming Fixation in Mathematics on Japanese Junior High School Students

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ABSTRACT

The purpose of this study is to find the influence of overcoming fixation in mathematics towards fluency and flexibility in open ended mathematics situation on Japanese junior high school students.

The students who can overcome fixation in mathematics can make varied ideas in open ended mathematics situation. The students who are unable to overcome fixation in mathematical thinking are able to solve similar questions in a short time.

1. BACKGROUND

Tammage[1] suggested that there is an urgent need for mathematics teachers identify, encourage and improve creative mathematics ability at all levels. He argues that mathematics teaching has, for too long, been dominated by a rational thought/route learning model, with an emphasis on cumulative learning of existing knowledge.

Haylock[2] created a framework for assessing mathematical creativity in school children. He identified two key aspects: the ability to overcome fixations in mathematics problem solving and the ability for divergent production within the mathematical situation.

Luchins[3,4] investigated the 'Einstellung' effect. Luchins's experiments centered on series of problems designed to establish a particular algorithm or stereotyped method of solution in the mind of the subject. In Luchins's famous Water Jugs Test, fixation is shown when the subject contiues to use the algorithm established in the first few problems in the series in later problems which have much simpler solutions.

A number of researchers considered the creativity in school mathematics as the notion of divergent production or divergent thinking, using parameters derived from the criteria used for assessing general creativity in divergent production tests[5].

Hollands[6] specifies creativity in mathematics and pays attention to divergent thinking in mathematics in terms of the following: fluency, shown by the production of many ideas in a short time; flexibility, shown by the student varying the approach or suggesting a variety of methods; originality, which is the student trying novel or unusual approaches; elaboration, shown by the extending or improving of methods; and sensitivity, shown by the student criticizing standard method constructively.

Assessment of creativity has conventionally been measured by divergent production tests, such as those established by Guilford[7,8] and Torrance[5]. Their tests contrasted divergent thinking with convergent thinking.

Dunn[9] reviews the work of a number of researchers who considered assessment instructions and constructs of divergent production in mathematics, mainly the United States. He found that it is good for measuring the ability of divergent thinking to use open ended problems in mathematics.

Silver[10] revealed that an open ended problem is good for students to develop creativity in mathematics. Students can write varied and original responses in an open ended situation in mathematics.

Evans[11] developed some problems measuring divergent thinking in mathematics and method of evaluation. He identified three aspects of divergent thinking in mathematics which were:fluency, flexibility and originality.

In this study, I will investigate the relationship between overcoming fixation in mathematical problem solving and divergent thinking (fluency and flexibility) in open ended mathematics situation in using the method of evaluation introduced and considered in Evans's research[11].

2. METHOD

2.1 Sample

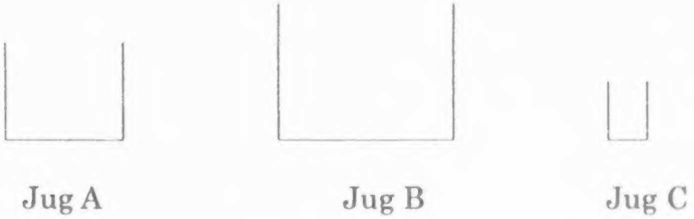
The sample consisted of students in two schools in Wakayama in Japan. The students were 11 and 12 years old in the 7th grade in junior high school. The sample consisted of 139 students.

2.2 Problems used by this investigation

The following are tests administered to the sample. These problems are introduced by Haylock's research[12,13].

<Problem (A)>

There are three jugs, A,B and C. Find the best way of measuring out a given quantity of water, using these jugs.



Example

Question Measure out 55 units.
Jug A holds 10 units.
Jug B holds 63 units.
Jug C holds 2 units Answer: B−A+C

Now, try these six questions. There is only one correct. Answer for each question.

Question 1 Measure out 52 units.
Jug A holds 10 units.
Jug B holds 64 units.
Jug C holds 1 unit. Answer:

Question 2 Measure out 14 units.
Jug A holds 100 units.
Jug B holds 124 units.
Jug C holds 5 units. Answer:

Question 3 Measure out 3 units.
Jug A holds 10 units.
Jug B holds 17 units.
Jug C holds 2 units. Answer:

Question 4 Measure out 100 units.
Jug A holds 21 units.
Jug B holds 127 units.
Jug C holds 3 units. Answer:

Question 5 Measure out 20 units.
Jug A holds 23 units.
Jug B holds 49 units.
Jug C holds 3 units. Answer:

Question 6 Measure out 5 units.
Jug A holds 50 units.
Jug B holds 65 units.
Jug C holds 5 units. Answer:

<Problem (B)>

The children in a class drew a sort of graph to show the number of boys and number of girls they each had in their families. When the graph was finished it looked like this:

* : 1 family

Girl(s)						
4	*					
3			*		*	
2	****	**	**	*		
1	*	****	****	*		
0		**	***	*	*	
	0	1	2	3	4	Boy(s)

Make up as many interesting and different questions as you can that can be answered from the graph.

Give the answers to your own questions, like this:

Question 1 How many families had 2 boys and 1 girl?
Answer 4

2.3 Procedure of analysis

<Problem (A)>

Student who write answers: (1)B–A–2C (2)B–A–2C (3)B–A–2C (4)B–A–2C (5)B–A–2C (6)B–A–2C in Problem (A), are one of the group of subjects. These students of this group can not overcome the fixation. These students' aspect of thinking as follows: as three or four answers are the same answers, the next answer will be the same answer.

Students who write answers: (1)B–A–2C (2)B–A–2C (3)B–A–2C (4)B–A–2C (5)A–C (6)C in Problem (A), are the other groups of subjects. These students of this group can overcome fixation. These students' aspect of thinking is as follows: though three or four answers are the same answer, the next answer may be a different answer.

<Problem (B)>

Problem (B) is the open ended problem, allowing the students to respond in a variety of ways. I calculated scores of three characteristics: fluency and flexibility. Fluency, in the research was defined as the flow of responses from the individual in the test. The measure of fluency is given by the number of responses. Hadamard[14] suggested that in mathematical discovery the starting of ideas can be more or less scattered, and the more the scattering, the more intuitive the mind. It is this scattering that is being sought by an examination of flexibility. The quality of flexibility refers to the variety of responses(question) in a given situation. For scoring purposes, all the responses given by the student are categorized with respect to certain criteria, and one point is given for each category represented in the student's set of responses. The criteria used for determining flexibility categories is as follow:

a) x: variable, y: variable
How many families had x boys and y girls ?

b) x: variable, y: variable, m: constant, n: constant
How many families had x boys > (or >) m boys and y girls > (or <) n girls ?

c) How many families had x boys and x girls ?
How many families had x boys and y girls in the case of x>y ?
How many families had x boys and y girls in the case of x<y ?

d) How many families had x boys and y girls in the case of x : y ?

e) z: variable
How many families had x boys and y girls in the case of x+y=z ?
How many families had x boys and y girls in the case of x–y=z ? or y–x=z ?

f) How many boys and girls are there in the case of x families in the graph ?

g) How many boys and girls are there in the case of x families > (or <) m families in the graph ?

h) How many families are there in the graph ?
How many boys are there in the graph ?
How many girls are there in the graph ?
Which is larger, the number of boys or girls ?

i) What is the mean number of children in a family ?
What is the mean number of boys in a family ?
What is the mean number of girls in a family ?

j) What kinds of patterns of boys and girls are there in the case of x families in the graph ?
How many kinds of patterns of boys and girls are there in the case of x family in the graph ?

k) What kind of family is the largest (or smallest) number of children (or boys, girls) ?

l) How many boys and girls do you have ?

Table 1 Significant difference of means in fluency and flexibility of divergent thinking between Group A and Group B

	Group A (n = 30)		Group B (n = 109)		
	Mean	SD	Mean	SD	t
Fluency	5.200	2.535	7.092	4.488	–2.198 *
Flexibility	2.033	0.983	1.275	0.588	5.270 **

Group A : Students who overcame fixation

Group B : Students who could not overcome fixation

Level of significance : ** 1% * 5% , n : The number of data

3. RESULT AND DISCUSSION

Students who overcame fixation in Problem (A) scored significantly higher in flexibility in Problem (B). Students who were unable to overcome fixation in Problem (A) received significantly higher scores in fluency in Problem (B), than students who overcome fixation.

Two conclusions may be drawn from this research. Firstly, the students who can overcome fixation in mathematics can make varied ideas in open ended mathematics situations. Secondary, students who are unable to overcome fixation in mathematical thinking are able to solve similar questions in a short time.

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A Case Study of Student Emotional Change Using Changing Heart Rate in Problem Posing and Solving Japanese Classroom in Mathematics

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This case study aimed to determine the intensity of emotion and to know the causes of it in Problem Posing and Solving lesson styles compared to Drill style. In order to achieve this aim, the study used heart rate to determine the intensity of two students' emotional responses and to identify the causes of emotional change during the lesson. Problem Posing lesson style showed the strongest emotional response. Five types of changing emotion were observed and the causes of emotional changes were explained from the social construction process of mathematical knowledge which was included in both Problem Posing lesson style and Problem Solving lesson style. Through this case study, the method of using changing heart rate for describing student emotional change was demonstrated.

Introduction

Several research studies have been done which focus on the relationship between the cognitive aspect and affective aspect (e.g. Plenary Panel in PME17 at Tsukuba). Using qualitative data from the teacher's viewpoint, some research has reported on how students represent their awareness of their emotional activity and what factors lead to change. Cobb et. al. (1989) discussed the important role of teachers to encourage students and teach them how they ought to feel in particular situations during mathematics lessons; a number of examples were given. Hannula (1998) gave an example of how a student represented her negative belief at a problem solving situation and how it changed in relation to her understanding of mathematics.

To demonstrate the effect of the problem posing and problem solving lesson styles[1] (Isoda,1992, 1996 and 1998), Isoda reported on positive changes in student emotion and belief through teaching. His first study (1992) used quantitative data to discuss evidence related to positive changes in students' belief in mathematics. To clarify the causes of positive change in process, student emotional response in the classroom should be documented using qualitative data. Isoda's second study (1994) used qualitative data to show that the particular lesson style gave students a special way of participating in the lesson and a chance for self-actualization in the mathematics classroom; the evidence was based on the results of facial expression analysis (Ekman, 1975). Using facial data, student emotional change related to social interaction can be observed. The result supports assessments by teacher in a classroom because observation of facial expression is a major way of a teacher's assessments in his teaching practice (e.g. Abe 1995). The difficulty is that a student's changing facial expressions during problem solving and problem posing by him/herself cannot be easily analyzed. Social interaction itself often occurs due to students' ideas about solving and posing, but during the process of solving and posing problems by themselves, students usually do not change their facial expressions much. A third study (1998) was designed to ask each student after each lesson to draw a graph of emotional change by him/herself (McLeod, 1990) which describe the changes of his/her purposes[2] (desires) during the lesson. The study used a pre-questionnaire, VTR and Worksheet in the lesson, a questionnaire and the graph of emotional change after the lesson, and an interview. The student's changing purpose within the classroom setting was well described: how he/she worked, how his/her own mathematical view (idea) developed, how he/she hoped to show it to others and why he/she wanted to compare his/her view with others. The results of the study showed that students engaged in a self-evaluation process as illustrated in Figure 1: the Value System includes Purposes, Context and Belief, specifying the object of evaluation and the value of evaluation; after self-evaluation, we detect an observable response which Cobb calls an *emotional act*. The result of self-evaluation usually led to the next context or purpose. Figure 1 is also implied from Mandler's view of arousal (1984) and McLeod's view of emotional change (1993). Affective processes are unobservable processes. The description of changes to the student's purpose was detected by the interpretation of data which included student's comments and the graph of emotional change after the lesson. The third study might describe the dynamism of the student's mind well; on the other hand, it looks somewhat like a re-writing drama using many kinds of qualitative data. It does not directly express the effect or dynamism of the lesson with the immediate intensity of the affective response as does live observation of the classroom.

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[1] Japanese styles usually use problems for teaching content. Posing, Solving and Negotiating are included.

Following on the studies discussed above, this case study aimed to measure how emotional intensity occurred throughout the lesson. In order to achieve this aim, the research used a heart rate (HR) monitor to show the intensity of the student's emotional response and identify the cause of emotional change in the context of lesson style. This study also demonstrates the method of using HR in research for mathematics teaching.

The reason why we used HR is that HR changes depending on emotional change in the same way as facial expression. From the result of Brain Science (Bloom, 1985), the limbic system that is a central part of emotional response also functions as the root for thinking and learning. From the working of the brain, HR changes directly depending on stress/tension, and speed is also controlled via the

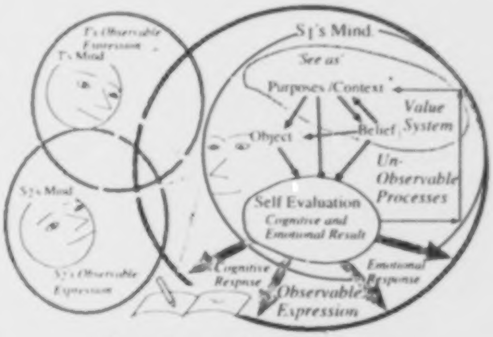

Figure 1. The model of Self-Evaluation

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[2] "I-Shi' in Japanese, but it means purpose, will, notion, intent, device, volition and desire in English.

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autonomic nervous system for keeping homeostasis. Thus HR is good quantitative indicator of the intensity of emotion and a good way for specifying the cause of changing intensity. However, HR also changes depending on physical work (Yamaji, 1985). Thus the cause of changing HR needs to be interpreted carefully.

Methodology

Because the aim of the study was to determine the effect of lesson styles, we observed students who had participated adequately with these kinds of lesson styles. Two observable fourteen years old (grade 8) students, student H and student N, were selected. Their achievements were average and they usually displayed a lot of emotional activity during lessons. Before the study, in order to describe their beliefs, they were asked to answer questionnaires that were selected from a number of research studies (IEA 1982, Itoho 1986, Schoenfeld 1989). Two hours of problem solving lesson style and two hours of problem posing lesson style were observed. For comparison, a drill-type computer tutorial (CAI) was observed for one hour as a control. In each lesson, H and N were recorded by VTRs and Worksheets and each of them put on an HR monitor (POLAR Vantage XL) that recorded their HR value at five second intervals each during the lesson. After each lesson, they were asked to draw a graph of their emotional change and observers asked them to explain their finished graph.

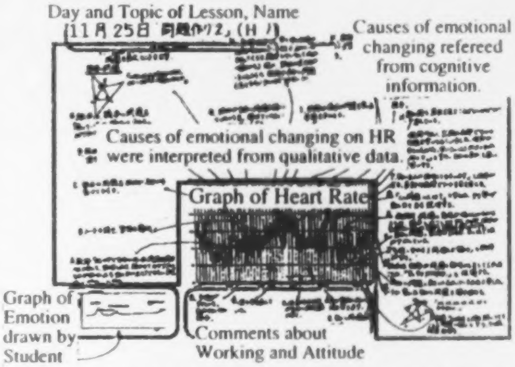
Day and Topic of Lesson, Name
11月25日 問題作り2, (H)
Causes of emotional changing refereed from cognitive information.
Causes of emotional changing on HR were interpreted from qualitative data.
Graph of Heart Rate
Graph of Emotion drawn by Student
Comments about Working and Attitude

Figure 2. Synthesized Description of HR Change and Activity

After the data were obtained, the HR data was entered into a spreadsheet and the graph of HR was drawn and compared with all of the other qualitative data. Unstressed (unloaded) H's HR was almost 77 beats per minute and unstressed N's HR was almost 68 beats per minute. The graph of HR represents intensity of changing HR. However, HR also changes just from breathing: in the case of students H and N, their HRs are fluctuating about six beats per minute from normal unstressed (at rest) breathing. Students' exertions (such as 'standing up') also contribute 'noise'. So the focus was directed to the phenomenon of HR changing by more than ten beats per minute, to interpretation of the meaning of changing HR using qualitative data (including interview), and to the careful exclusion of 'noise' such as that due to physical exertion resulting from student activity. The interpretation was synthesized into one diagram per one student for one lesson. Figure 2 is an example of a synthesized diagram at the lesson 2 of the problem posing lesson style.

Results

Figure 3 displays the quantitative differences among Drill style (control), Problem

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Solving style and Problem Posing style. Italic t-values mean significant differences. These differences between styles in each student should explain the difference of their value system based in Figure 1. Indeed from the graph of emotional change drawn by H, H described his bad image with respect to Drill on CAI. N described his enjoyment of Drill because his score on Drill was very good, but did not describe good feelings with respect to lessons of the Problem Solving style.

Figure 3. Quantitative Differences among Lesson Styles

	Drill on CAI	Prob Solving 1	Prob Solving 2	Prob Posing 1	Prob Posing 2	
Num counted	540	554	576	408	588	Counted 5 sec.
H's Average	86.76	92.47	92.53	102.3	98.62	Beats per Min.
H's Variance	16.87	37.01	43.07	168.5	60.17	
t value	Control	*3.49*	*3.17*	*2.89*	***5.046***	
N's Average	82.97	74.39	82.36	88.57		Beats per Min.
N's Variance	11.23	15.22	33.90	18.32		
t value	Control	*10.7*	0.444	*5.93*		

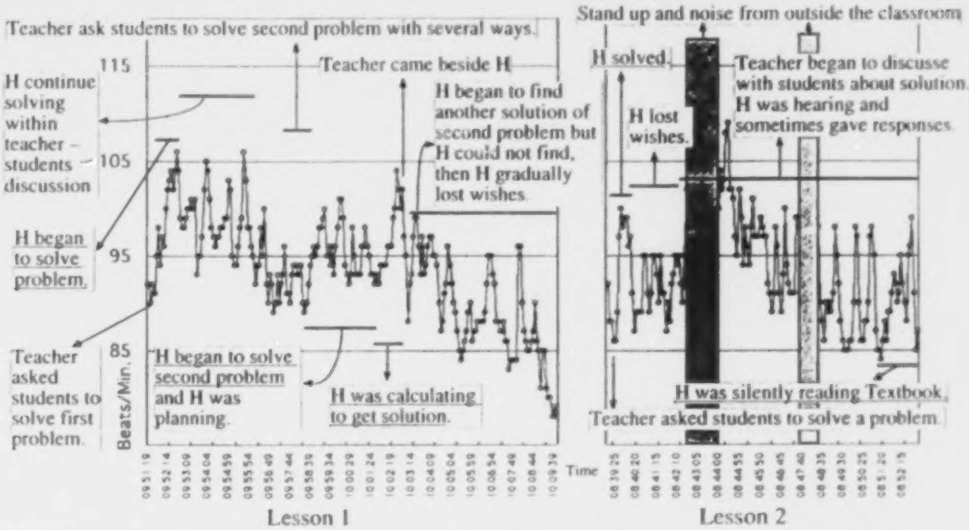
Dark columns were omitted because the HR monitor malfunctioned by **battery down**.

If the t value is larger than 2.596, the level of significance is less than 0.5 percent for a one tailed test.

The order of the average of H's HR is the Problem Posing lesson style, the Problem Solving lesson style and the Drill style. For N, the Problem Posing lesson style is also the highest. With the Problem Posing lesson style, the student designs problems based on his/her ability and interest (Value System). Following problem design, students would show something of their personality to others through their problems, and develop an appreciation of other students' problems and something of their personality. On the other hand, with the Problem Solving lesson style, the teacher usually asks students to solve given problems designed for teaching new content which emerges through solving and negotiating. In the case of the Drill lesson style, CAI responds immediately with the student's score, but there is weak social interaction. From the result of figure 3, we can say that H's and N's value system was mostly oriented towards the Problem Posing lesson style and secondarily, H was oriented towards the Problem Solving lesson style while N was not so clear cut in this case study.

We noted above that there was a difficulty in fitting the student's value system in the Problem Solving lesson style. Figure 4 shows parts of lesson 1 & 2 (Problem Solving lesson style) for student H. In lesson 1, the teacher gave students unknown problems and both of H & N described their enthusiasm. In lesson 2, the teacher gave students an easy problem and both of H & N reported that they lost the desire to think about it because they thought they already knew the problem solution. Indeed in Lesson 1 of figure 4, H was engaging in solving for about 15 minutes until he could not find another solution and at the time of negotiation (omitted in figure 4), H's HR was also changing a lot. On the other hand, in Lesson 2 of figure 4, H was engaging in solving for about one minute only. After he lost the desire to think, H was only listening to discussion between the teacher and other students until eventually he began to read

Figure 4 Parts of Lesson 1 & 2 by Problem Solving Lesson Style



his textbook silently. The Problem Solving lesson style was not invoked in the lesson 2; this is reasonable as this style is not needed if students believe that they already know the solution.

By interpreting the HR graphs with qualitative data, we distinguished the following five types of changing emotional intensities which were commonly observed in both H and N: *Rapid and strong intensity of changing HR* was observed in social interaction (the student's ideas are examined in a social context such as someone else asking for an explanation of his/her idea, the student wanted and expected to explain the idea to others and a similar, better or counter idea was explained by other students). *Rapid intensity of changing HR* was also observed at the stage of beginning to think and in many cases, it was *increasing gradually* if the student could continue thinking and was getting a better perspective or view of the situation by thinking. *No changing of HR* or *rapid decreasing of HR* was observed when the student does not need to think about the situation any more.

Figure 5 is an example of H's changing HR and episodes from lesson 2 of the Problem Posing lesson style, corresponding to the last 27 minutes of figure 2. The evidence of *Rapid and strong intensity of changing HR* are shown in italics in figure 5. At (1) in figure 5, the teacher moved to M, who was next to H, to read M's problem and H expected that the teacher must be reading H's problem. At (2), the teacher took H's worksheet and at (3), began to discuss it with H. At (5), the teacher asked H to write his problem on the chalkboard in order to explain it to other students. At (9), the teacher asked four students including H to explain their problems. At (10), H was planning his explanation while other students were explaining their problems. At (11), when the teacher asked H to begin his explanation, H's HR recorded a peak

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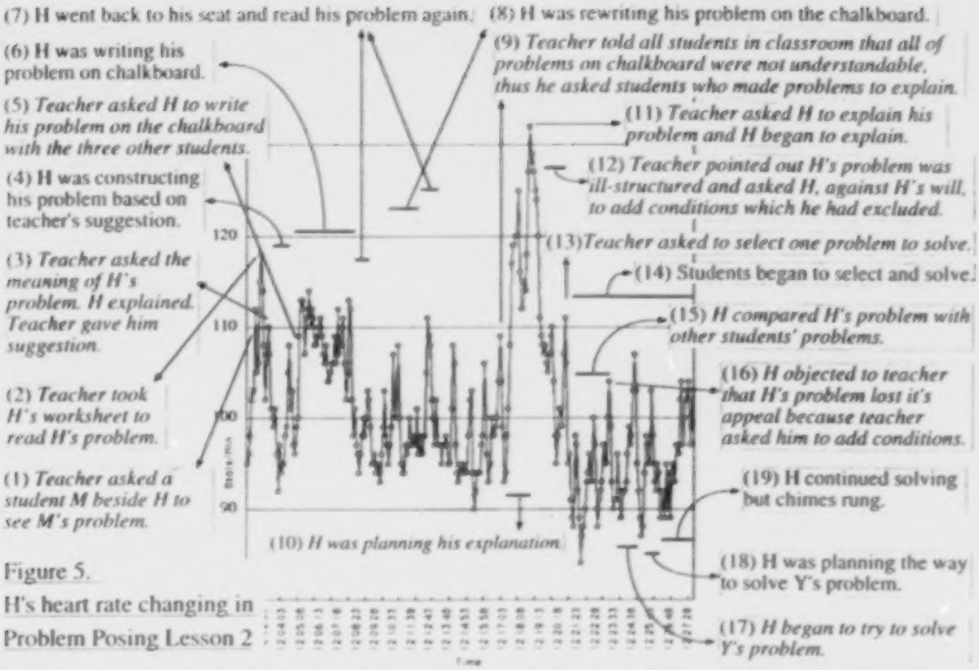

Figure 5.
H's heart rate changing in Problem Posing Lesson 2

at 132 beats per minute. At (13), the teacher asked students to select a problem to solve, and H then compared his problem with others. H expressed an objection to the teacher at (16) regarding his own problem, and he recognized Y's problem to be a nice problem at (17). All of these episodes are related to H's idea being examined by others. *Rapid intensity of changing HR* and *HR increasing gradually* are shown in episodes[3] (4), (10), (15), (17), (18) and (19) of figure 5 as well as the underlined boxes in figure 4 which involve solving, posing, planning or reading. During these episodes, H's HR gradually increased for more than one minute. In figure 4 and figure 5, *rapid decreasing of HR* was usually observed after *HR increasing gradually* and *rapid & strong intensity of changing HR*. We observed H's *no changing of HR* during the Drill style while he did not engage in it.

Discussion

In an ERIC search[4] by words 'heart rate' from 1966, there are studies using mathematical or arithmetic problems which looked at stress or anxiety but with the aim of research into disease, disruptiveness, physical exercise or so on. One study discussed the effect of anxiety related to item order in a mathematics test (Hambleton,

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[3] (7) of figure 5 must be also but the changing HR at the left part of (7) is 9 beats per min. less than ten.

[4] In an ERIC search by words 'heart rate', 117 articles were hit in the 1990's. Most studies used HR in relation to physical exercise. Some studies used HR in relation to psychological or emotional problems such as the effects of a behavior therapy and the intensity of stress about a specific situation such as a presentation.

1968). Research which uses HR for describing the student's emotional change in relation to mathematics teaching could not be found. This study has particular relevance in demonstrating how HR can be used to describe a student's emotional change depending on lesson practice in the mathematics classroom. Indeed, the results show that a student's changing HR is interpretable using other qualitative data in classroom and the result of this interpretation implies that changing HR can be expressed in terms of the arousal of the student's mind which can be related to his/her value system.

The first feature of this study involves interpreting the intensity of HR as the positive emergence of belief toward the lesson style. Indeed, before this study, H and N already had a positive view of the mathematics classroom: for example, in the pre-questionnaires, they stated that they liked mathematics and learning. Even their anxiety[5] to explain their own idea to others is a positive result of belief depending on the lesson style, because it implied that they knew the aspects of the lesson style and we should teach students how to enjoy each lesson style.

From the model in figure 1, their belief about the lesson style could be seen from their changing HR. The most intense emotional change came from their need to present their idea in the classroom; they were asked to explain their idea, they wanted and expected to explain their idea to others and a similar or better idea was explained by other students. Their positive belief in thinking about mathematics problems and designing mathematics problems also emerged with strong emotional intensity. The gradual increasing of HR represented their concentrated thinking and the graph of emotional change drawn by the students explained it as their expectation of getting across their own view about the task. At the same time, we should know that it shifted to their anticipation of presenting their idea during the lesson. Thus we can say that their emotion changes depending on the social construction cycle of mathematical knowledge (eg. Ernest, 1994). This is the second feature of this case study because the cycle is implied by data including physiological HR data. These first and second features strongly depend on experimental design because the two students selected had already participated positively in these kinds of lessons.

The graph of emotional change drawn by the students after each lesson did not represent changing emotional intensity as measured by HR, but it nevertheless implied the social construction cycle. The graph drawn by the students was usually represented as a smooth curve. From the graph of changing HR, we can say that many things were omitted which the student was not conscious of or could not remember or did not feel important or did not desire to report. On the other hand, the change of HR expressed the moments of emotional changes during each lesson. From the comparison of both graphs, we can find what the student had omitted and we can specify the reason using other qualitative data. The results about their emotional changes shown by HR enabled a new interpretation of the graph of emotional change as drawn by the student. This is the third feature of this study, because it shows how the method of changing HR as used in this study and the method involving students to drawing a graph of their emotional change (McLeod, 1990) mutually support the interpretation of student's emotional change.

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[5] Anxiety usually includes negative nuance. Mandler preferred Stress to avoid negative or positive nuance.

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USING STUDENTS' STATISTICAL THINKING TO INFORM INSTRUCTION

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Abstract

This study designed and evaluated a teaching experiment in data exploration for a grade 2 class. The teaching experiment was informed by a cognitive framework that described elementary students' statistical thinking. Following the teaching experiment, the children showed significant gains on all four statistical processes associated with the framework. Case-study analysis revealed that: experiences with different kinds of data reduced children's idiosyncratic descriptions, categorical data was more problematic than numerical data; technology was helpful in stimulating children's ability to organize and represent data, children possess prior conceptual knowledge of center and spread; context was important when children analyzed data.

In response to the critical role that data play in our technological society, there have been world wide calls for reform in statistical education at all grade levels (e.g. School Curriculum and Assessment Authority and Curriculum and Assessment Authority for Wales, 1996; National Council of Teachers of Mathematics, 1998). These calls for reform have advocated a broader approach to the study of data handling, one that includes describing, representing, and interpreting data (Shaughnessy, Garfield, & Greer, 1996). Notwithstanding these recommendations, there has been relatively little research on elementary students' statistical thinking and even less research on the efficacy of instructional programs in data exploration. For instance, some elements of students' statistical thinking have been investigated (Aberg-Bengtsson, 1996; Curcio & Artzt, 1997; Bright & Friel, 1998; Gal & Garfield, 1997; Mokros & Russell, 1995; Pereira-Mendoza & Mellor, 1991), but research has not developed or used the kind of cognitive models that researchers like Fennema et al. (1996) deem necessary to guide the design and implementation of instruction.

This study addressed that void by developing and evaluating a teaching experiment on data handling with young children. More specifically the study sought to: (a) use a cognitive framework that describes students' statistical thinking to design and implement a teaching experiment with grade 2 children and (b) evaluate the effect of the teaching experiment on students' learning.

Theoretical Considerations

The teaching experiment used in this study was adapted from Cobb (1999). The design phase of the teaching experiment involved the planning of instructional sequences or hypothetical learning trajectories (Simon, 1995) that were informed by research-based knowledge of students' thinking. More specifically, drawing on the theory of cognitively guided instruction (Fennema et al., 1996), we used a Framework (Jones et al., 2000) (Figure 1) that describes elementary students' statistical thinking

Process/ Level	Level 1 Idiosyncratic	Level 2 Transitional	Level 3 Quantitative	Level 4 Analytical
Describing Data Displays (D)	•gives a description that is unfocused and includes idiosyncratic/irrelevant information; has no awareness of graphing conventions (e.g., title, axis label(s)) of the display •does not recognize when two displays represent the same data OR indicates some recognition but uses idiosyncratic/ irrelevant reasoning •considers irrelevant/subjective features when evaluating the effectiveness of two different displays of the same data	•gives a description that is hesitant and incomplete, but demonstrates some awareness of graphing conventions •recognizes when two different displays represent the same data, but uses a justification based purely on conventions •focuses only on one aspect when evaluating the effectiveness of two different displays of the same data	•gives a confident and complete description and demonstrates awareness of graphing conventions •recognizes when two different displays represent the same data by establishing partial correspondences between data elements in the displays •focuses on more than one aspect when evaluating the effectiveness of two different displays of the same data	•recognizes when two different displays represent the same data by establishing precise numerical correspondences between data elements in the displays •provides a coherent and comprehensive explanation when evaluating the pros and cons of two different displays of the same data
Organizing and Reducing Dat (O)	•does not group or order the data or gives an idiosyncratic/ irrelevant grouping •does not recognize when information is lost in reduction process •is not able to describe data in terms of representativeness or "typicality" •cannot describe data in terms of spread; gives idiosyncratic / irrelevant responses	•gives a grouping or ordering that is not consistent OR groups data into classes using criteria they cannot explain •recognizes when data reduction occurs, but gives a vague/irrelevant explanation •gives hesitant and incomplete descriptions of data in terms of "typicality" •invents a measure--usually invalid--in an effort to make sense of spread	•groups or orders data into classes and can explain the basis for grouping •recognizes when data reduction occurs and explains reasons for the reduction •gives valid measures of "typicality" that begin to approximate one of the centers (mode, median, mean); reasoning is incomplete •uses an invented measure or description which is valid, but the explanation is incomplete	•groups or orders data into classes in more than one way and can explain the bases for these different groupings •recognizes that data reduction can occur in different ways and gives complete explanations for the different reductions •gives valid measures of typicality that reflect one or more of the centers; reasoning is essentially complete •uses the range or invented measure that has th same meaning as range
Representing Data (R)	•constructs an idiosyncratic or invalid display when asked to complete a partially constructed graph associated with a given data set •produces an idiosyncratic or invalid display that does not represent or reorganize the data set	•constructs a display that is valid in some aspects when asked to complete a partially constructed graph associated with a given data set •produces a display that is partially valid, but does not attempt to reorganize the data	•constructs a display that is valid when asked to complete a partially constructed graph associated with a given data set; may have difficulty with ideas like scale or zero categories •produces a valid display that shows some attempt to reorganize the data	•constructs a valid display when asked to complete a partially constructed graph associated with a given data set; works effectively with scale, zero categories. •produces multiple valid displays, some of which reorganize the data
Analyzing and Interpreting Data (A)	•makes no response or an invalid/irrelevant response to the question, "What does the display not say about the data?" •makes no response or gives an invalid/incomplete response when asked to "read between the data" •makes no response or gives an invalid/ incomplete response when asked to "read beyond the data"	•makes a relevant but incomplete response to the question, "What does the display not say about the data?" •gives a valid response to some aspects of "reading between the data" but is imprecise when asked to make comparisons •gives a vague or inconsistent response when asked to "read beyond the data"	•makes multiple relevant responses to the question, "What does the display not say about the data?" •gives multiple valid responses when asked to "read between the data" and can make some global comparisons •tries to use the data and make sense of the situation when asked to "read beyond the data;" reasoning is incomplete	•makes a comprehensive contextual response to the question, "What does the display not say about the data?" •gives multiple valid responses when asked to "read between the data" and can make coherent and comprehensive comparisons •gives a response that is valid, complete, and consistent when asked to "read beyond the data"

Figure 1. Statistical Thinking Framework

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to envision how learning might proceed and how it might be supported pedagogically. These instructional sequences were subject to review and modification in light of experiences gained during the teaching experiment. The classroom-based analysis phase also used the Framework to interpret classroom events and to examine changes in the children's statistical thinking. Although we took some cognizance of classroom social factors (Cobb), the main focus of our study was on student learning.

The Statistical Thinking Framework. The Framework (Figure 1) incorporates four key data handling processes adapted from Shaughnessy et al. (1996): describing, organizing and reducing, representing, and analyzing and interpreting data. Describing data involves explicit reading of data contained in a visual display and recognition of graphical conventions (Curcio & Artzt, 1997). Organizing and reducing data incorporates mental actions such as ordering, grouping, and summarizing data using measures of center and spread (Moore, 1997). Representing data involves the construction of visual displays including displays that exhibit different organizations of data. Analyzing and interpreting data includes what Curcio and Artzt referred to as "reading between the data" and "reading beyond the data" (p. 124). The Framework descriptors for each statistical process build on previous research in describing data (e.g. Beaton et al., 1996), in organizing and reducing data (e.g. Bright and Friel, 1998), in representing data (e.g. Zawojewski & Heckman, 1997), and in analyzing and interpreting data (Curcio & Artzt, 1997). For each of these processes, four levels of thinking were hypothesized and validated (Jones et al., 2000). Level 1 is associated with idiosyncratic thinking, Level 2 is transitional between idiosyncratic and quantitative thinking about data, Level 3 involves the use of informal quantitative thinking, and Level 4 incorporates analytical and numerical thinking about data. These levels of thinking are consistent with neo-Piagetian theories that postulate the existence of levels of thinking that recycle during developmental stages (Biggs & Collis, 1991).

Method

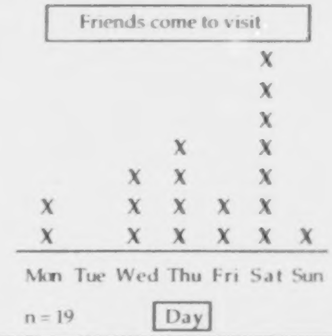
Participants. The participants in the study were the grade 2 students, the mentors, and the researchers. The class of 19 students involved in this teaching experiment was located in an elementary school in central USA. In addition to the analysis involving all students, four students working in pairs (pseudonyms: Jena and Ryan and Cher and Kirsten) were purposefully chosen for case-study analysis. The students were chosen according to mathematics achievement--2 from the middle 50%, and 1 from each of the upper and lower quartiles. Ten education majors, in their junior or senior year of undergraduate study acted as mentors for pairs of students. Each mentor had a dual role of facilitating the instructional sequence and observing the thinking of a pair of students. The first two researchers acted as witnesses, one for each targeted pair of students and their mentor.

Procedure. The intervention phase of the teaching experiment comprised nine 40-minute sessions, two sessions per week over 5 weeks. Sessions opened with a whole class exploration led by the second author and then mentors worked with student pairs in accord with the instructional sequence designed by the research team. In the week prior to the first session of the intervention, each of the 19 students was assessed using an interview protocol based on the Framework. The procedure was repeated for the postintervention assessment in the week following the intervention.

Instructional sequence. The instructional sequence comprised a series of problem tasks, data sets, and follow-up questions focusing on the four statistical processes of the Framework. The problem tasks and data sets were situated within the context of a Butterfly Garden project that the class was undertaking. Each pair of students had access to Graphers (Sunburst Communications, 1996); a software package for primary grades that enables students to enter, organize, and represent data. Following each session, the research team reviewed the classroom events especially those that occurred in the two pairs targeted for case-analysis. Where necessary, revisions were made to the learning trajectory. Mentors participated in weekly sessions to discuss ways to: (a) establish social norms for working in pairs, (b) pose data-exploration tasks, (c) use the Framework to assess and build on children's thinking, and (d) enhance collaboration and negotiation of shared meanings.

Data collection, instrumentation and analysis. Interview and observation data were gathered from three sources: (a) researcher-designed interview protocols, (b) mentor evaluations of children's statistical thinking during each session, and (c) researcher-witness narratives on each pair of target students. The interview protocol comprised 39 items (9 on describing, 12 organizing and reducing, 3 representing, 15 analyzing and interpreting) in three different contexts: How Many Friends Came to Visit? Beanie Babies and The Beanbag Game. Selected items are presented in Figure 2 and each question is labeled D, O, R, or A according to which Framework process is assessed.

Sam had some friends come to visit each day during one week in the summer. The number of friends and the days they visited were displayed like this:



D1 What does this picture tell you?
A3 Which day had the highest number of visitors? Which day had the lowest number of visitors?
A4 How many friends came to visit Sam during this week?
O1 About how many friends came to visit Sam each day?
O2 What is the average number of friends that came over to visit Sam each day?
R1 Can you draw this graph in another way?
A5 About how many friends would you expect to come to Sam's place every week during the summer vacation?
A6 How many friends would you expect to visit Sam during a 4-week month?

Figure 2. Selected items: Sam's friends

A double coding procedure (Miles & Huberman, 1994) was used to establish pre and postintervention statistical thinking levels for all 19 students on each of the four processes. In this procedure, the first two authors independently coded, according to the levels of the framework, all questions on each student's protocol. The modal response level for each statistical thinking process was used to determine a student's dominant thinking levels. Where the response data was bimodal the median was used. The authors reached agreement on 127 out of 152 levels, that is 84%. Differences were clarified until consensus was reached. For further details see Jones, et al. (2000). Both "within" and "cross-case displays" (Miles & Huberman, 1994, pp. 90, 207) were used to guide the analysis of qualitative data from the 4 target students. Summaries of mentor evaluations and researcher-witness field notes were coded and synthesized to discern learning patterns exhibited by target students during the intervention. A Wilcoxin Signed Ranks Test (Siegel & Castellan, 1988, pp. 87-95) was used to compare pre and postintervention statistical thinking levels of the 19 students.

Results

The effect of the teaching experiment: Quantitative analysis. A Wilcoxin Signed Ranks Test (Siegel & Castellan, 1988) revealed significant differences between the pre and postintervention thinking levels of the 19 students on each of the four statistical thinking processes: describing data ($p < .001$); organizing and reducing $p < .001$), representing data ($p < .002$), and analyzing and interpreting ($p < .004$). Table 1 shows the frequencies of statistical thinking levels for the four processes prior to and following the intervention. The most salient feature of the data was that the number of students exhibiting level 3 thinking increased following the intervention and this was accompanied by a corresponding decrease in the number of students exhibiting level 1 thinking. The growth was more noticeable for the first three processes, with the trend for analyzing and interpreting data being similar but not as pronounced.

Table 1: Frequency of Statistical Thinking Levels for Each Process

	Levels				
Frequency	1	1.5	2	2.5	3
Describing Data Displays (D)					
Preintervention	6	2	10	0	1
Postintervention	1	0	8	3	7
Organizing and Reducing Data Displays (O)					
Preintervention	7	1	10	0	1
Postintervention	0	0	12	1	6
Representing Data (R)					
Preintervention	7	0	10	0	2
Postintervention	1	0	9	3	6
Analyzing and Interpreting Data (A)					
Preintervention	8	3	7	1	0
Postintervention	2	2	11	2	2

The effect of the teaching experiment: Case study analysis. A number of learning patterns and trends were discerned by examining the relationship between target students' learning during instruction and their thinking at the pre and postintervention assessment points. These learning patterns are described and interpreted for each of the four statistical thinking processes.

With respect to *describing data*, Jena and Ryan's initial responses exemplified the kind of idiosyncratic or cosmetic thinking that was typical of Level 1 students. When asked about the line plot for Sam's friends (D1, Figure 2), Jena read the title and related the content to her own situation. Even though she recognized the names of the days she made no attempt to state how many friends visited on particular days. However, experiences with both categorical and numerical data during the intervention seemed to focus her thinking and produce less idiosyncratic descriptions. For example, in the postintervention assessment, Jena not only specified the number of friends who came to visit Sam each day she also stated " x is a friend," in essence, abstracting her own legend. Jena, Cher, and Kirsten all showed greater facility in dealing with graphical conventions and the data as a whole by the end of the intervention. Interestingly, categorical data was more troublesome for these children than numerical data. Children's intuitive thinking with respect to *organizing and reducing data* was problematic. Although all of the target children except Jena were able to organize the Beanie Baby data in the preassessment interviews, they were unable to reorganize the categorical data in the butterfly project using paper and pencil. Technology proved helpful in stimulating their sorting schemas and Cher and Kirsten invented their own way of reorganizing the data by literally dragging data values across the desktop. Once again target students performed better with numerical data like age and time of arrival than they did with categorical data like school attended. Prior to and during the intervention children showed some conceptual knowledge of mode and middle in dealing with center and some idea of "density" in dealing with spread.

In the preassessment interviews, all of the children had difficulty in *representing data* on Sam's Friends in a valid alternative way (R1, Figure 1). However, Jena, Cher, and Kirsten were able to complete an unfinished representation of this data. Their limited accessibility to sorting schemas constrained their thinking in relation to representing data and during the intervention we also noted that they didn't think of the data as a single entity only as individual data values. Once again using technology to represent data helped stimulate their mental processes and it enabled Cher to build her own graph on the desktop in preference to using the menu of graphs provided by the software. Interestingly it was Cher's partner Kirsten who exhibited the highest thinking level in the postassessment. With respect to *analyzing and interpreting data*, children's thinking prior to the intervention seemed to be more normative on tasks that involved reading between the data than on tasks that involved reading beyond the data. For example, all target children were generally able to deal with questions like A3 or A4 (Figure 2). However questions on reading beyond the data (e.g., A5, A6, Figure 2) were difficult for all children. During the intervention we found that children's thinking in relation to reading the data was not as complete as we had thought. For example, Jena and Ryan weren't able to compare the number of students who attended the Butterfly Garden before and after 1 p.m. We don't believe that students were incapable of ordering the data before and after 1 p.m., rather it seemed that they were unwilling to focus on subsets of the data "prior to 1 p.m." and "after 1 p.m." Notwithstanding this unanticipated problem, Jena and Ryan came up with a very well reasoned discussion on a butterfly life-cycle question that involved reading beyond the data. Familiarity of the context seemed to be a key factor in their response.

Discussion

Quantitative analysis on the performance of all 19 children in the teaching experiment showed that they made significant gains on all four statistical processes between the pre and postintervention interviews. The students made their greatest gains in describing data displays-- a result that is consistent with previous research (Beaton, et al., 1996, Pereira-Mendoza & Mellor, 1991). Our study revealed that even though students initially seemed to focus only on surface features (Lyons, 1977), extended exposure to different data sets (within the Butterfly Garden context), resulted in stronger connections between the meaning of a data display and its specific features and conventions. With respect to organizing and reducing data and representing data, the children's limited sorting schemas were an impediment to organizing data and ipso facto to constructing visual displays. Consistent with the findings of Hancock et al., (1992) for older students, we found that technology was an effective instructional vehicle for stimulating children's visual representations. Given earlier findings of Mokros and Russell (1995), the children's diverse conceptual knowledge of center and spread surprised us. However further research is needed to determine how best to use this prior knowledge in instruction. Children's thinking in analyzing and interpreting data, especially reading between and reading beyond the data (Curcio & Artzt, 1997), was inconsistent. While reading between the data seems more accessible, their thinking was dependent on context familiarity and it took time to build this familiarity even in our study where all data focused on the Butterfly Garden theme. Consistent with research on older students (Bright & Friel, 1998), we also found that the children had difficulty seeing data as a total entity or focusing on subsets of the data.

Given the prior knowledge and growth that children demonstrated on all four statistical processes, there is evidence that children can accommodate a broader approach to data exploration. However, if instruction on data exploration is to reach its full potential in the elementary grades, there is a need for further research to build learning trajectories that link different levels of children's statistical thinking.

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The Student Experience of Online Mathematics Enrichment

インターネットを利用した生徒の数学に対する見方に関する研究

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Following concerns about the falling number of mathematics majors at University level, consideration has being given in a number of countries to enhancing provision for those school pupils who show the potential to study mathematics at University. Prevalent amongst this provision are enrichment opportunities and material designed to provide a wider picture of mathematics. This paper reports findings from a study of internet-based enrichment material in mathematics and focuses on the student experience of such online provision. Data from questionnaires and interviews suggest that this enrichment material helped the pupils who used it to gain a wider appreciation of mathematics and raised the profile of mathematics as a subject that could be interesting enough to pursue beyond school. Issues of equity in access to the material remain.

本稿は、インターネット教材*nrich*について、特に生徒がこのページを通してどのような経験を得たかということに焦点を当てて考察したものである。アンケート及びインタビューにより得られたデータから、このホームページは、生徒に数学のよさを理解させることと、学校で教えられる教科としての枠を越えた数学の側面に対する見方を助長させることができるということが示唆された。今後の課題としては、このページへのアクセスの公平さに関することである。

Introduction

In a number of countries and over a number of years, demand for undergraduate units in mathematics has increased as rising numbers of undergraduates are required (or elect) to study an element of mathematics as a component of their degrees. At the same time, the number of students choosing to *major in mathematics* has fallen, or, at best, has remained static despite a substantial increase in the total number of students entering University. In the US, for example, the number of mathematics majors peaked in the early 1970s and is currently about the same as it was 40 years ago (estimate based on figures in Madison 1990). The US Committee on the Mathematical Sciences in the Year 2000 (1991 p2) report that "interest in majoring in mathematics [in the US] is at an all time low". Similar disquiet has been expressed in the UK (London Mathematical Society 1995, National Committee of Inquiry into Higher Education 1997) and in many other parts of the world (Leder *et al* 1998).

Concerns such as these have led to efforts to enhance the provision of mathematics for those school pupils with the potential to study the subject at University. In the

UK, the needs of the more able pupil have recently been highlighted by a government-funded research review (Freeman 1998) and a parliamentary report (House of Commons Education and Employment Committee 1999). This paper reports findings from a study of enrichment material in mathematics provided through the internet (Jones and Simons 1999). The data indicate that use of this enrichment material had an impact on the beliefs the pupils held about mathematics. The pupils gained a wider appreciation of mathematics and the profile of mathematics as a subject was also raised. It became more of a subject that could be interesting enough to study beyond school. Yet the data also revealed that the majority of users of the material were white boys and that a large proportion accessed the material from home, giving an indication of the likely socio-economic status of their families.

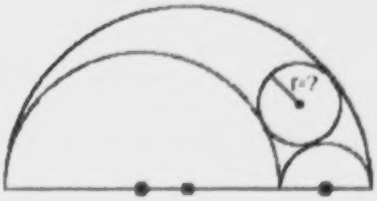
Theoretical Framework and Related Research

Studying mathematics can evoke strong emotions in pupils. In a comprehensive review, Schoenfeld (1992 p359) concludes that pupils "abstract their beliefs about formal mathematics .. in a large sense from their experiences of the classroom" and further that "students' beliefs shape their behaviour in ways that have extraordinarily powerful (and often negative) consequences". Beliefs that doing mathematics solely means following the rules laid down by the teacher and that knowing mathematics only means remembering and applying the correct rule allow no space for essential attributes of mathematical activity such as creativity and problem-solving. Such beliefs are implicated by Schoenfeld as explanatory factors for the ever decreasing proportion of students who wish to study mathematics as a main subject the further up the education system they progress. Malmivuori and Pehkonen (1996) report how beliefs impact on pupil achievement in mathematics and Brown (1995) illustrates the influence of teachers on children's image of mathematics. Further reports of research on the impact of pupil beliefs in mathematics can be found in Pehkonen (1996).

Curriculum enrichment "is the deliberate rounding out of the basic curriculum subjects with ideas and knowledge that enable a pupil to be aware of the wider context of a subject area" (Freeman 1998 p44). It is an approach that is consistently advocated for the more able (see, for example Koshy and Casey 1997) and has been recommended in various ways in the teaching and learning of mathematics over a number of years (see, for example, House 1987, Kennard 1996, Sheffield 1999). The 'NRICH online mathematics enrichment project' was established in the UK in 1996 to promote an interest in mathematics and to assist the mathematical development of children who have the potential to go on to study mathematical subjects at university. The principle method of meeting these aims is through the provision of regular online mathematics 'magazines' containing puzzles, problems and games, enhanced by mathematics undergraduates acting as peer-teachers providing an electronic answering service for learners. An example problem, most suitable for older school pupils, perhaps older than age 15, is shown overleaf. The problem is from the November 1999 edition of the online magazine.

Just touching[1]

Two semi-circles are drawn on one side of a line segment. Another semi-circle touches them externally as shown in the diagram. What is the radius of the circle that touches all three semi-circles?



Further details of the project and access to the current edition of the online magazine and to the NRICH archive can be found on the NRICH website: http:\\nrich.maths.org.uk

In what follows we report some of the findings of a study of how the use of the NRICH website facilities impacts on the mathematical development of children who have the potential to go on to study mathematical subjects at university.

Methodology

The overall design of the study incorporated a range of methods. For the component of the study reported in this paper, the principle method was a questionnaire completed by pupils who accessed the website during May 1999. In addition, visits were made to a sample of schools, and interviews were conducted with a range of pupils (some 20 in all). The methodology reflected the technology-based nature of the NRICH project. The questionnaire was web-based, a suitable method as the target population was well-defined (Schmidt 1997), and some pupil interviews were conducted through e-mail correspondence (Roselle and Neufeld 1998). Standard techniques were adopted to develop and test the questionnaire with piloting of both a paper and electronic version (Oppenheim 1992, Dillman 1999). The questionnaire covered type of access, form of usage of NRICH material, and evaluative comment on the material. The questionnaires also sought information on the type of school and household (or other location) where NRICH material was accessed. Respondents could also offer to be contacted again by e-mail to provide follow-up information. Respondents were assured of the confidentiality of all the data. Full details of the methodology, including copies of the questionnaire and interview schedules, can be found in Jones and Simons (1999).

Analysis of data

We begin with data from the questionnaire collected during May 1999. As was anticipated, a number of incomplete or frivolous responses to the questionnaires were logged and so close scrutiny was paid to the data in order to ensure the validity of the data set used for analysis and hence the reliability of any conclusions drawn from the analysis. Data sifting and analysis followed standard procedures to ensure no bias was inadvertently introduced. Following these procedures, 199 pupil questionnaire responses were accepted for analysis.

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[1] Copyright of NRICH material is held by the University of Cambridge. Permission is granted to print and copy NRICH material on paper for non-commercial use.

The analysis of the questionnaire data revealed the following:

- pupils from 15 different countries accessed the NRICH website during May 1999
- nearly 60% of the pupils who accessed NRICH were from the UK (Australian pupils were the next highest category, with over 95% of them attending private schools)
- over two-thirds of the pupils were boys
- almost two-thirds were white (the next highest ethnic group was Chinese, the majority being from Singapore and Australia)
- almost three quarters were of secondary school age (aged 11-16)
- the pupils attended a range of institutions, with the largest number, although just less than 30% of the total, attending secondary comprehensive schools (pupils from private preparatory schools came a close second with most of these being Australian)
- over half of the pupils were recommended to try the NRICH website by their teacher (browsing the net and information from a parent were the other main ways of finding out about NRICH)
- around half of the pupils accessed NRICH at school while almost exactly the same proportion accessed NRICH at home - there was no difference in the pattern of place of access between boys and girls. Only a handful accessed NRICH from a public library or other public access location.
- for most pupils there was no 'maths club' provided as an extra-curricular facility at their school
- the majority of pupils accessed NRICH about once a month. They mostly accessed the pages of mathematical problems and puzzles. There was little difference between the relative usage of NRICH by girls and boys.
- pupils had positive views about the facilities provided by the NRICH project with almost half saying that NRICH was better than the mathematics they did in school
- most pupils thought that NRICH had made them more interested in mathematics and more likely to continue studying mathematics

Virtually all the comments made by pupils on the questionnaires were complimentary about the NRICH project. Some typical examples were:

> *"All the Nrich problems are challenging to me, not like the problems I do at school. Doing the* [NRICH] *problems has made me feel how it feels to be stuck on a mathematics problem and do not know how to do it."*
> 11 year old boy from Singapore.

> *"I really like this site and think it is great, although my friends don't like maths that much I am trying to wean them onto this site because it is really interesting and helps me with my maths."*
> 12 year old girl from England.
>
> *"I think N-rich is really cool and has made me think differently about mathematics."*
> 12 year old boy from the USA.
>
> *"NRICH is cool. It does not just do the simple types of mathematics but it does problem solving as well. It is really fun and is great to log onto during breaks."*
> 12 year old girl from a private upper school in England.

All the pupils interviewed for this study (some 20 in all), either in school, or using e-mail correspondence, were complimentary about the NRICH project facilities. They invariably said that the mathematical problems on the NRICH website enlivened sometimes routine mathematics lessons or gave them interesting things to think about during breaks or lunch-time. Almost all the pupils interviewed had submitted solutions to NRICH and very much liked seeing their solution published on the NRICH website. Making the website more interactive so that problems could be solved 'online' was one suggestion made that would improve the site in the eyes of the pupils.

Other evidence, from the school visits and from the questionnaire completed by 450 teachers (full details in Jones and Simons 1999), shows that pupil exposure to NRICH problems and puzzles (in schools where teachers accessed the site) was likely to be wider than the pupils were aware of. Teachers reported accessing the NRICH website to use it as a source of interesting mathematical problems to enhance their regular classroom teaching. Indeed, one teacher admitted that:

> *"The reason that I do not recommend NRICH to my pupils is that it is an important resource for me to use in the class."*
> Teacher in an English suburban primary school.

Of course the majority of teachers did say that they had, on occasion or perhaps regularly, recommended NRICH to their pupils. Teacher recommendation tended not to be restricted to more able pupils in mathematics but was used more widely. Nevertheless, the majority of the teachers using NRICH felt (many strongly) that NRICH was particularly good for pupils who had a talent for mathematics.

The majority of teachers questioned thought that using NRICH had made their pupils more interested in mathematics, although around 20% said that they did not know. Evidence for this improvement was generally in terms of increased interest in mathematics shown by pupils and by pupils accessing the NRICH website of their own accord. In general the teachers thought that NRICH enhanced the pupil view of mathematics by regularly providing novel and interesting problems that often afforded a new way of approaching a standard school mathematics topic.

Summary and Discussion

Evidence from this study suggests that pupils using the NRICH website facilities gained by having access to interesting mathematical problems. For some pupils, these mathematical problems were more stimulating than the mathematics they regularly did at school. Many pupils who accessed NRICH did so from home which is an indication that the NRICH materials are intriguing enough to attract pupils in their own time. Some pupils accessed NRICH quite frequently, another indication of the quality of the materials. Only a minority of pupils made use of either the bulletin board facilities available through the NRICH website or the answering service. Those that did so spoke highly of the service and how it stimulated further thought. These pupils particularly valued the opportunity of being able to ask mathematical questions and receive replies. Seeing their solutions published on the NRICH website was also popular with pupils.

Girls were under-represented as NRICH pupil users. Certain ethnic groups from the UK might also have been under-represented (such as pupils of black Caribbean and Pakistani heritage) but the numbers of respondents was not sufficient to draw any firm conclusions. Data on the socio-economic class of pupils was not collected as it is notoriously difficult to collect such data accurately and reliably but the large proportion of pupils who accessed NRICH at home is one indication of the likely socio-economic status of their families. Few pupils accessed NRICH through a public library or other public access location.

The main impact of NRICH on the pupils who accessed the site was in terms of helping them to gain a wider appreciation of mathematics and raising the profile of mathematics as a subject that could be interesting enough to pursue either within or outside school or for further study. Quantifying this impact was beyond the scope of this study. The web-based nature of the NRICH project was also an important factor and was associated with the functionality and accessibility of the NRICH website which was judged by both pupil and teacher users to be well-designed.

Concluding comments

Research indicates that the beliefs about mathematics held by pupils can be deep-rooted and difficult to influence (Schoenfeld 1992, McLeod 1994). Hannula (1998) reports, in a case study of one particular pupil, that change can happen as a result of the pupil experiencing different learning materials and classroom approaches. The evidence from the NRICH project supports the idea that enrichment material designed to provide a wider picture of mathematics can encourage pupils to view mathematics as a subject that could be interesting enough to pursue beyond school.

According to data from the Third International Mathematics and Science Study (Beaton *et al* 1996 p 126) around 80% of 12 and 13 year old pupils in England like mathematics at school while only about 5% say that they dislike mathematics "a lot" (comparative figures for Japan, for example, are 53% and 11%). Over 90% of this age group of pupils in England thought that they usually did well in mathematics (this compares to 44% in Japan). It is possible that this positive disposition to mathematics of pupils in England is a factor in making them open to the possibility of viewing mathematics as an interesting subject, one worthy of consideration for further study beyond school (as part of TIMSS, pupils were not specifically asked if they found mathematics interesting nor whether they were envisaging studying mathematics at University).

In another component of the TIMSS study, teachers were asked what particular abilities were very important for pupil success in mathematics (Beaton *et al* 1996 p 142). When asked about the importance of pupils being able to think creatively, teachers in England were next to the bottom of the list of countries with just over 30% saying that it was a very important factor (only France, out of the 41 countries, was lower with 20%). This lack of stress on creativity in mathematics by teachers may also be a factor in influencing the impact of enrichment material, such as that available through the NRICH project, which encourages a view of mathematics as a creative subject, on pupils' views of mathematics.

While the impact of the NRICH material was found to be positive on those pupils that accessed it, issues of equity in access to the material remain. The ability to access the internet is growing both in schools and in homes but the distribution of access is not equitable across income groups. The study reported in this paper found that most pupil users of NRICH were white boys and that a large proportion of NRICH users accessed the site from home. Access from public libraries and other places of public access was very low. Efforts need to be made by Government agencies in order to reach those categories of users currently under-represented so that all may benefit from access to material such as that provided by NRICH.

If projects such as NRICH are to be fully successful in encouraging more students to continue with mathematics, it appears that the experience of students in studying mathematics at University would also benefit from some attention. Seymour and Hewitt (1997) report how, in addition to fewer undergraduates choosing to major in mathematics, a disturbing number of mathematics majors switch to other courses during their University careers. Amongst the reasons they found for this 'switching' was loss of interest in the subject, the belief that other subjects were more interesting, poor teaching and the feeling of being overwhelmed by the pace and load of curriculum demands. While projects such as NRICH are showing their value, the scale and nature of the issues to be tackled in ensuring that pupils experience mathematics as an interesting and rewarding subject of study requires the collaboration of all those involved.

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EPISTEMOLOGICAL FEATURES IN THE MATHEMATICS CLASSROOM: ALGEBRA AND GEOMETRY

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Abstract. *The organisation of the mathematical content by the teacher and its evaluation and interpretation by the pupils influence the attempt of the latter to make sense of the mathematical ideas. This article investigates the emergence of the epistemological features in two rather distinct areas of the secondary school mathematics curriculum: algebra and geometry. The results of the analysis show that the teaching practices tend to "level" the epistemological differences between the two contexts reported in the literature.*

1. Introduction: epistemological features in the mathematics classroom

For pupils, mathematics acquires its meaning through its school content as well as the relationships and interactions developed among the social and cognitive patterns of the classroom. The organisation of the elements of the content, their interrelations, their relationship with problems, situations and representations and the evaluation and interpretation of these elements within the classroom and the individual's functional role inside or outside the classroom play an important role in pupil's attempt to make sense of school mathematics (Sierpinska and Lerman, 1996, Steinbring 1991). Children learn what is important in mathematics by observing the elements emphasised and the types of answers rewarded by the teacher and fellow pupils. It could be argued that pupils interpret the classroom events by attaching to each of them a value proportional to its usefulness to the mathematics lesson. These interpretations concern the meaning of the concepts and processes (the content of the mathematical knowledge) as well as their nature and value in the system of the mathematical knowledge (epistemological features of mathematics).

In this context, the study of the nature and the organisation of the mathematical content within the classroom acquires a particular importance. Such a study requires the analysis of the ways in which the *epistemological features*, that is, the *nature, meaning and definitions of the mathematical concepts, the theorems (properties) and the solving, proving and validation procedures emerge in the mathematics lessons.* The relevant literature is very limited with the epistemological studies of Heinz Steinbring providing a framework of thinking on this issue (e.g., Steinbring, 1997).

In an earlier study focused on the teaching practices of mathematics in the primary school, we found that teachers deal with the epistemological characteristics of the subject (mathematical content) in a unified manner (Ikonomou et al., 1999). That is, the way in which the epistemological features of mathematics are presented and dealt with in the primary classroom does not allow them to be distinguished from one another with respect to their meaning and function in mathematics. In an attempt to interpret these results, we argued that this might be seen to reflect primary school teachers' limited knowledge of the subject matter.

In a consequent study (Kaldrimidou et al., 1999), we extended our sample to the early secondary years in order to investigate whether a broader knowledge of the subject matter by the teachers (secondary teachers are mathematicians in the Greek educational system) would make a difference in the way in which the epistemological features of mathematics emerge in the classroom. The results of the study showed that the organisation of the mathematical knowledge in the secondary classroom does not bear characteristics compatible to the epistemological features of mathematics: concepts and definitions are reduced to processes of manipulation, construction and recognition and theorems are not differentiated from definitions and processes.

The above considerations give rise to an interesting question: how this homogeneity is realised in the context of the two main branches of the school mathematics curriculum, i.e. of algebra and geometry, *given that these two areas differ epistemologically and promote rather distinct patterns of thinking.*

2. Research on school algebra and geometry: epistemological issues

In the context of the school mathematics, algebra is usually introduced as generalised arithmetic (generalisation of patterns and the laws governing numerical relations) and is often described as «the transition ... accompanied by processes of evaporation of meaning, which have to be converted into processes of condensation...This, in turn, requires ..an epistemological transition from a procedural to a relational perspective» (Arzarello, 1998). Although other ways of introducing algebra have appeared over time, generalisation, abstraction and the use of a «manipulative symbolic language to aid this work» (Bell, 1996) are seen to be at the heart of algebraic thinking. Thus, the teaching of algebra aims to develop ability in the recognition of structures, the expression of symbolic relations and operations on these expressions through reasoning which is sequential, logical and verbal.

Kieran (1992) distinguishes two aspects of the school algebra curriculum: the procedural (arithmetic operations carried out on numbers) and the structural (operations carried out on algebraic expressions) which parallel Sfard's operational (as a process) and structural (as an object) conceptions of a mathematical notion respectively «whereas the structural conception is static, instantaneous and integrative, the operational is dynamic, sequential and detailed» (Sfard, 1991). Both Kieran and Sfard agree that for most people, the operational conception comes first and it usually requires a lengthy period, whereas the transition to the structural needs more explicit attention than it currently receives in mathematics teaching. The relevant literature shows that the difficulties children encounter with algebra can be seen to reflect the failure of many pupils to access the structural component of school algebra. This is usually interpreted as a result of school practices in algebra which over-emphasise the operational aspect, and offer very little support in facilitating access to the structural conception of algebraic ideas (there is usually a rapid move to meaningless symbolic manipulations).

As far as school geometry is concerned, many studies have shown the difficulties pupils encounter with the theorisation of the geometrical objects and relationships necessary to introduce them to more advanced mathematics, such as, for example, the axiomatic foundations of Euclidean geometry, the proving processes and space geometry (Hershkowits, 1990, Berthelot and Salin, 1994). These difficulties are often attributed to the emphasis placed in school upon the visual aspects of the subject matter and the pupils' confusion between geometric drawings and geometric objects. These constitute obstacles to their understanding of the basic geometrical concepts and lead children to dealing with the geometrical objects in a visual-perceptive rather than relational – analytical way (Boero and Carruti, 1994, Duval, 1995, Hershkowits et al., 1996).

Most of the recent studies in the area focus on the search for appropriate teaching situations and teaching environments to overcome the above difficulties and develop processes of generalisation, theorisation and proof in geometry. These environments (the micro-worlds being a prime example of such an environment), which impose a different treatment of the geometrical shapes, allow the pupils to make links between visual and theoretical aspects of geometry (Laborde, 1993, Marrioti, Bartolini Bussi, 1999).

In this paper, an attempt is made to investigate the way in which the epistemological features of mathematics are treated in the above drawn framework of school algebra and geometry as well as the degree to which this contributes to the pupils' problems and difficulties with the acquisition of the relevant ideas. In this investigation, the requirements of each of these two branches of school mathematics are taken into account.

3. The study

The data reported here come from a rather large project which focuses on the teaching of mathematics in the nine years of the Greek compulsory educational system, as shaped by the National Curriculum, the textbooks, the teachers' views and practices and the pupils' mathematical knowledge as assessed by the teachers. The study aims to investigate the possibility of applying alternative mathematics teaching approaches in the Greek school.

The research problem addressed here focuses on the emergence of the epistemological features of mathematics in the context of the secondary school algebra and geometry and its relationship to the development of relevant ideas by the pupils. In particular, the study attempts to examine the following research questions:

- does the management of the epistemological features of mathematics in the context of the teaching of the school algebra and geometry elevate the special characteristics of each of these two areas of the school curriculum?
- how does the treatment of the epistemological features of mathematics in the two contexts contribute to the acquisition of relevant ideas by the pupils?
- does the management of the mathematical knowledge in the two contexts have a role in pupils' difficulties with the relevant ideas?

The data come from 24 different secondary classes (13–15 years old) from 12 different schools in 3 geographical areas of the country: one urban, one agricultural and one semi-urban. In each class, two mathematics lessons were observed in different times and on different topics; these were video–taped and transcribed. Teachers were strongly advised to "work as usually". It is worth noting that all Greek schools use the same textbooks, distributed by the Ministry of Education.
The transcripts were analysed with respect to the organisation and interrelationships of the various elements of the mathematical content (concepts, definitions of concepts, theorems and functionality of theorems).

4. Presentation of the data and discussion

In the following, episodes from various lessons are used to illustrate the findings. These episodes include elements of the epistemological features of the mathematical content. The focus of the analysis is on the features concerning the mathematics "produced" in the classroom rather than teachers' or pupils' epistemology.

The analysis showed that *definitions* and *theorems* are often confused with processes. The mathematical function of a definition and *its differentiating and identifying* role are rarely elevated in the classroom. Instead of defining concepts, teachers tend to simply explain to the pupils how to manipulate the elements to which the definitions refer. A definition is frequently asked to "be learnt" to satisfy the demands of a specific lesson and is then abandoned. In the same way, theorems are memorised as rules of action. In particular, definitions and theorems are reduced to processes of manipulation in algebra and of visual recognition or of drawing in geometry.

Example A1-algebra. The teacher starts by defining factorisation mathematically.
T(eacher). *The factorisation of an algebraic expression consists of its transformation to a product of two or more other algebraic expressions.*
(A little later the mathematical method turns into a process)
T. *We will study about ten cases, will see them one-by-one and we will learn practical rules.. so, if we are given this, we will do that, and so on..*
(In the end of the lesson, the two cases studied become rules)
T. *Let me make one or two observations: we notice that when the powers of the same letter appear in all the terms of the polynomial, then the power of this letter with the smallest index comes out of the bracket. The second case concerned the grouping of the terms .... The common factors of each group come out of the bracket and what remains inside the bracket in each group is the same.*
In this episode, although the mathematical definition of factorisation is initially provided, it is never activated. When it comes to actually factorising, there is no discussion of the meaning of the common factor. Thus, the mathematical knowledge does not become the object of the negotiation of meaning, but it is presented as a set of ready-made instructions of a factual character (on symbols rather than on specific numbers). Furthermore, the emphasis is placed upon the morphological elements of the transformation of the algebraic expression, as indicated by the formulations: "this letter with the smallest index comes out" and "what remains inside the bracket".

Example A2 - algebra. Pupils attempt to factorise the expression x^2+2x+3. The teacher presents the discriminant as a manipulative procedure, which makes possible the factorisation of the quadratic expression.

T. *Are there any two numbers with sum of 2 and product of 3? Try ... You cannot.. Therefore, the other method we are going to learn will safely tell us if this quadratic expression can be analysed. Write in your books what I write on the board:* $D=b^2-4ac$.

(They substitute and calculate the result: -8)

T. *This is less than zero. When this quantity is less than zero, you should know in advance that this expression cannot be turned into a product. All right? While you are now thinking, «how can we understand that it can't be analysed», since we cannot find two numbers with sum of 2 and product of 3, this quantity called discriminant comes along to tell us that we cannot. Following the other way, we are hesitating, whereas this way we are 100% sure.*

The discriminant is given with no explanation as to its meaning and its relation to the roots of a quadratic equation. Such an explanation would make it easier for the pupils to understand that the factorisation is impossible, since there are no real roots (the square root of a negative number doesn't exist in the set of real numbers). Thus, the whole process would have become meaningful and wouldn't be encountered as a calculating instruction and a rule of inference.

Example A3-algebra. Pupils try to factorise the expression 7(a-b)-a+b. The peculiarity of the signs leads the teacher to ask for the special process children use in similar cases.

T. *What do you notice here? What is common?*

P(upil). *a-b*

T. *So, we notice that these terms are exactly the same as these terms* (shows)*, but what is the difference? Their signs, isn't it? And what do we do, when the signs bother us?*

P. *We take out the minus sign ...*

T. *We take out the minus sign and we close them into a bracket with the signs we want, with the opposite signs outside.*

In this example, we have again clear indications of morphological recognition: the minus sign is taken out.

Example A4-algebra. The teacher confirms the answers to his questions.

T. *Can we analyse* a^2+2?

P. *No*

T. *Why?*

P. *Because it is a sum*

T. *Correct, because it is a sum (!)*

The teacher bases his elaboration on morphological elements of the algebraic expression ("+" sign) and not on the nature of the terms (squares) and their property (the square root of a negative number does not exist in the set of real numbers).

Example G1- geometry. The teacher tries to define the distance of a point from a line. Facing pupils' inability to understand, he quickly switches from the definition to the drawing procedure.

T. *How do we define the distance of a point from a line? Tell me. Distance of a point from a line. Anyone? Use your own words...*

P. *It is the vertical line segment which determines the distance between two points*

T. *Any better?*
P. *It is the perpendicular segment which we draw from the line..*
T. *From the point* ... (he realises the difficulty the pupils encounter). *Who would like to draw a line on the board? Come to the board, take the ruler, draw a line and a point A wherever you like.. Draw a distance* (the pupil draws a perpendicular). *Which one is the distance?* (the pupil shows) ...
(They try another case and the teacher asks again for the distance)
P. *The distance of a perpendicular from the point A to the point B.*
T. *Up to point B, but in a perpendicular manner, eh? How would you place the ruler here?*
P. *We place the right angle on the line and* ... (he does so)
Neither the teacher nor the pupils feel any need to define or pay any attention to the point B (the intersection point of the perpendicular with the given line). This underlines the visual – perceptive, rather than the relational – analytical, approach adopted in relation to the mathematical knowledge under consideration. The negotiation of meaning declines to a manipulative instruction, with no explicit reference to the nature of the points.

Example G2- geometry. The teacher starts by giving the definition of an altitude of a triangle. However, the definition is virtually "destroyed" in the following, as the whole lesson is based on the drawing, focusing on the way in which the altitude is to be drawn.
T. *We first give the definition. What is the height of a triangle: we call height of a triangle the distance of a vertex from its opposite side.*
(A child repeats. The teacher works on the drawn shape)
T. *So, how are we going to place the ruler, here, watch ...The one side will go through the point (vertex) and the other (should be placed) on the side. We will do it practically, take the (right-angled) ruler..*
As in the previous example, the definition is reduced to a manipulative step-by-step instruction on the actual drawing.

Example G3-geometry. The class works on the angles between parallel lines intersected by another line. The teacher asks the pupils to decide which angles are equal. One child attempts to formulate the relevant theorem, but the teacher directs the whole effort to the immediate recognition of the properties on the drawing.
P. *When two parallel lines are intersected by ...*
T. *No, shall we say it in a simpler manner? We decide which angles are acute and which ones are obtuse and then we know which ones are equal ...*
In this example, it is clear that concepts and relations between the geometrical objects "are substantiated" by the drawing, while the perceptive-morphological elements predominate. At the same time, explicit instruction is provided which identifies this approach to conceptualisation as an acceptable one in mathematics.

Example G4-geometry. The class works on finding the symmetrical points in an axis and in a centre symmetry. The visual aspect is dominant and the teacher encourages, almost "imposes", the visual recognition of the geometrical objects by the pupils.
T. *We need a geometrical shape to apply some properties to find the symmetrical point.*
P. *Here we have a rectangle*
T. *Which one? We have many here.*

P. *It is ...*
T. *Come on the board to show us, we have used letters, but you better show it to us.*
P. (he stands on the board and shows, failing to use letters to symbolise). *This is because the opposite sides of the rectangle are equal..*
T. *Correct. Which side with which?*
P. (always showing) *This is equal to this and this is equal to this.*

As in the previous example, there is a move towards recognising from the drawing rather than from the geometrical objects and their relations.

5. Concluding remarks

Algebra and geometry differ epistemologically and underline different patterns of thinking and learning. A number of studies concerning the teaching and learning of school algebra highlight the importance of pupils' passage from the operational to the structural conception of algebraic ideas. The difficulty reported is often attributed to the teaching practices reducing the algebraic objects to symbolic manipulations -that is, teaching practices tend to treat the algebraic entities as arithmetic ones. In school geometry, on the other hand, research pinpoints a dichotomy between the concrete, visual nature of a drawing and the abstract, analytical-relational and general nature of a shape. Children's difficulties are usually interpreted on the basis of this dichotomy which leads, and at the same time is attributed, to a reconnoitring, pictorial elaboration of the geometrical objects and relations.

Considering the preceding examples from these two perspectives, we ascertain that the way the mathematical knowledge is handled in the secondary classroom is concordant with the basic findings of the relevant literature reported above. Furthermore, it also appears to *foster reconnoitring and morphological elements in algebra and handling/manipulative elements in geometry.* This suggests that the management of mathematical knowledge in the two contexts does not only prevent the differentiation of their epistemological elements (homogeneity of mathematical elements of different epistemological meaning), but moreover it unifies them (homogeneity of mathematical objects of different epistemological origins). In other words, the epistemological differences between the two areas reported in literature disappear in the classroom.

This homogeneity adds to pupils' difficulties in their attempt to construct the mathematical meaning for themselves. Everything appears to be the same: objects and their relations are not separated. The usage of representational-morphological criteria and calculating manipulation of the objects and relations provokes a downgrading and suppression of the nature of mathematics and meaning, and creates serious learning obstacles, since according to the literature it does not constitute an appropriate approach. In addition, their usage in all areas of school mathematics intensifies the confusion caused by this lack of differentiation, thereby further reinforcing pupils' faith in inappropriate approaches, as these are suggested and often imposed by the teacher.

The role of the teacher in this organisation of mathematical knowledge is decisive, as the examples illustrate. A possible explanation can be sought in teachers' interpretations of their leading role in teaching mathematics as well as their attempts to manage mathematical meanings through means which are more accessible and comprehensible to the pupils (drawings, measurements, rules, and so forth).

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WHAT DO WE REMEMBER WHEN IT'S OVER?
ADULTS' RECOLLECTIONS OF THEIR MATHEMATICAL EXPERIENCE

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This paper reports on a qualitative study focusing on adult point of view in regard to high school mathematics. A two-part individual interview was conducted with 24 men and women between the ages of 30 to 45, all engaged in successful careers. The subjects were requested to reflect upon themselves as learners of mathematics in the past and as consumers of mathematics in the present. The interviews were analyzed within a framework called 'the personal mathematical profile', designed during the research. Profiles were grouped into main types that are described here along with some complementary comments.

What is the most commonly heard question in high school math classes? To the best of our knowledge, no research has ever addressed this intriguing issue directly. Nevertheless, on the basis of personal experience only, we venture to guess that the leading candidate is: *Why do I have to learn this*? Indeed, we speculate that most math teachers confront this question, and some of them even attempt to seriously answer it. Such an attempt forces one to deal with what might be called 'the gap between declarations and adult-life reality'. This gap is a well-known one (Boaler, 1999). On the one side stand all the worthy targets that as teachers we can identify with, targets derived from living in a rapidly changing world, as expressed in the famous NCTM standards (NCTM, 1989). On the other side we find the constantly heard arguments about the irrelevancy of most high school mathematical content to adult life, and the need to rethink the curriculum. An assertion of this kind is made, for example, by Burke (1990):

> "Math appears to be everywhere, but it is usually in the form of arithmetic or statistics – not abstract algebra or triangles. [...] It is time to call a halt to specialized high school math requirements and not take math teachers' words for what the rest of us need" (p. 15,17).

Both sides, it seems, share the same anticipation: young people should benefit from their school years by acquiring valuable knowledge. Different interpretations of the word 'valuable' might lead to different questions and thus to various research approaches via which to pursue the answers. One can regard 'valuable' as 'useful' or 'applicable' and therefore focus on real-life situations in order to explore the kind of mathematics involved in them. The work of Lave (1988), Nunes et al. (1993) and Hoyles et al. (1999) exemplify such socio-cultural investigations.

The notion of 'valuable knowledge' might also be perceived as 'lifelong enrichment'. This interpretation evokes the issue of memory, an issue traditionally addressed by cognitive psychologists. How much of the content studied in school becomes what

Bahrick (1979) calls permanent knowledge? Due to understandable methodological problems, relevant research can hardly be found (see a review by Semb et al., 1993). In regard to mathematics, a unique work is that of Bahrick and Hall (1991), a quantitative study using multiple regression techniques to examine the effects of several variables on lifetime maintenance of high school mathematics.

We would like to offer another way to construe the phrase 'valuable knowledge'. In our view, it is the kind of knowledge that one does not regret having learnt. Knowledge can be valued because it is useful, but not necessarily so; even if it is not utilized, even if it is not well remembered, still it was worth acquiring. This perception embeds affective components of learning, as well as practical and cognitive ones.

In the study presented herein we were interested in the following questions: How do educated people view their mathematical experience in high school? Do they feel that they acquired valuable knowledge, in the sense described above? What are their opinions about the need to study mathematics, in light of their maturely-modified retrospection? Finally, to what extent can they recall basic mathematical concepts and procedures?

Methodology

Two major décisions preceded and shaped the choice of method. The first was to focus on adults with considerably high level of education. The implied assumption was that the voice of subjects holding college or university degrees might define "the upper limit" of people's views on mathematics education.

The second decision was to concentrate on in-depth qualitative analysis of data collected from a relatively small number of subjects. Undeniably, the combination of qualitative methods and statistical analysis relating to a larger sample could contribute to a more complete picture (Carrol, 1999). However, priority was given to the qualitative method, bearing in mind the possibility of future research that will build upon the results of the present one as a base for a large-scale investigation. The present study can therefore be defined as a "collective case study" (Stake, 1994).

Subjects were selected from the population of a small village near Jerusalem. This village was founded some fifteen years ago, and the adult residents - approximately 200 in number – came from all over the country and had attended different high schools in their youth. Members of this community can be seen as representing the higher-educated level of the society in Israel. Restricting the study to volunteers between the ages of 30-45 who were not mathematics graduates or math teachers led to a group of 105 candidates. Three factors were then considered: gender, level of math taken in high school[1] and current profession[2]. After mapping these characteristics of the candidates in a sampling table, subjects were chosen randomly from each cell. The final number of subjects was 24 (12 men and 12 women), all graduates of college or university and engaged in professional careers in areas such as law, medicine, psychology, art, business, high school teaching and other.

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[1] In Israel mathematics is a compulsory subject throughout high school, and can be studied in three levels, herein referred to as high level, intermediate level and low level.

[2] The candidates' professions were categorized by The Standard Classification of Occupations, a scale of 10 categories published by the Israeli Central Bureau of Statistics. It is noteworthy that 83% of these 105 professions were classified within the three top categories.

Each participant was separately interviewed. The interview was semi-structured and consisted of two parts. During the first part, several general questions were posed to the subject, allowing the conversation to evolve in directions led by his or her responses. Subjects were requested, for example, to describe a typical high school math lesson as they remembered it, and recollect their feelings at the time; to assess the impact of studying mathematics on their lives in comparison to other high school contents; to suggest a future state policy in regard to mathematics education; to elicit free associations to the words "professional mathematician", and more. In the second part of the interview, subjects were asked to solve mathematical tasks involving basic concepts and procedures. Analysis of this part will be presented elsewhere. Each interview lasted between 2 to 3 hours. The interviews were recorded and transcribed.

Theoretical framework

Transcripts of the first part of the interviews were read and analyzed within a framework developed by a process known as categorical-content analysis (Lieblich et al., 1998). We named this framework 'The Personal Mathematical Profile'. The profile consists of two sections: 'mathematical self-schema' and 'general opinions towards mathematics', as explained below.

Mathematical self-schema. The term is derived from the wider notion of self-schema (Markus, 1980), defined as "the cognitive structure that contains generic knowledge about the self" (Brewer, 1986, p. 27). We defined a mathematical self-schema, accordingly, as a specific structure containing the knowledge a person has of oneself as a learner and a user of mathematics. When analyzing a transcript, the following components were searched for as indicators of this schema:

1. The nature of feelings expressed by the subject when recollecting the experience of studying math in high school.
2. Characteristics of mathematics regarded by the subject as attractive or repellent.
3. Self evaluation of mathematical ability.
4. The relevancy ascribed to mathematics by the subject, in relation to his or her professional and everyday life.
5. The subject's involvement in the mathematical studies of his or her children.
6. The nature of feelings that were elicited while the subject was dealing with mathematical tasks during the second part of the interview.

General opinions towards mathematics. Apart from knowledge about oneself, one is likely to hold personal ideas concerning mathematics in regard to the public. The second part of the profile refers to these general opinions. Here we considered answers given by subjects, whether directly or indirectly, to the following questions:

a. What attributes are needed in order to succeed in mathematics?
b. Should mathematics be a compulsory subject throughout high school?

c. How important is mathematical knowledge as part of "being a well-educated person"?
d. What kind of mathematics is useful in adult life?

Results

Based on the components 1-6 and a-d above, a two-section profile was composed for each participant. Then, examining similarities and differences between the twenty-four profiles, 4 major types of mathematical self-schema and 4 patterns of general opinions were formed. Given below is a brief description of these types and patterns.

Types of mathematical self-schema.

I. *The positive schema.* Among the features of this schema are feelings of pleasure, challenge and superiority associated with studying mathematics in high school, along with high self-confidence in regard to mathematical ability. Mathematics is perceived as an appealing and intellectually gratifying pursuit, as it requires understanding rather than memorization. Subjects demonstrating this schema appeared to have a mathematical cast of mind[3]. They willfully use numbers in their professional lives as well as in their everyday surroundings. They enjoy calculating in their heads, even when a calculator is available. They are actively involved with their children's mathematical education. Most of them looked amused when asked to recall some mathematical material. While trying, the word 'fun' was sometimes used, along with expressions of embarrassment and disappointment when recalling failed.

II. *The semi-positive schema.* This type of schema is characterized, amongst other things, by feelings of respect towards mathematics. High school math is remembered as an important topic, a top priority goal that could not be easily achieved. Subjects identified with this schema described themselves as highly motivated former math students. They had worked hard, not always with pleasure, in order to succeed in mathematics. Most of them were attracted by the need to reason logically, but lacked the patience to accurately follow necessary procedures. Today they perceive themselves as rational people, able to handle mathematical content but not remarkably talented in this area. They regard their children's math studies as very important, but active involvement takes place only if the spouse is considered less fit for it. All of them value their mathematics education, mostly because it contributed to their reasoning skills. In this sense, they find the mathematical knowledge per se to be of secondary importance and, apart from basic arithmetic and geometry, irrelevant to their adult life. When requested to deal with recalling tasks, they tended to express mainly feelings of embarrassment and doubt about their ability to perform well. As the process proceeded, most of them were quite eager to reach the correct answer.

III. *The indifferent schema.* This schema is essentially defined by its title. Subjects exhibiting this schema described themselves as indifferent math students in high school. They were not enthusiastic, nor frustrated by it. They simply went through the motions of this requirement. This sort of recollection is not associated with low self-confidence in regard to mathematical competence; on the contrary, most of these subjects claimed to have a reasonable mathematical ability. It was motivation they lacked. Efforts in mathematics were given low priority, after other school and out-of-school occupations. Math was considered a boring, technical subject, with rare gleams of appealing ideas. Today, these subjects are still indifferent towards mathematics, though some of them said they assume it has appealing sides, only unknown to them. They all use calculators regularly in the course of their workday, but relate to the mathematics that is relevant to their lives as minimal and trivial. They sometimes assist their children with homework in math, but only as part of their parental duties; most of them said that if math became optional in high school, they would not insist on their children studying it. Attempts to recall the requested mathematical material elicited mainly feelings of discomfort and some expressions of inferiority.

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[3] A term used by Krutetskii (1976) for the inclination to view the world mathematically.

IV. *The negative schema.* Key words for this schema are fear, resentment, stress and frustration. Subjects identified with this schema described math lessons in high school as the worst part of their studies. Capable students at other areas, they found mathematics to be difficult and unattractive, a tedious list of exercises to be carried out, usually not very successfully. This situation was ascribed by different subjects to one or more of the following causes: lack of mathematical talent on their own part; lack of empathy and positive approach on the part of a key teacher, a behavior which was described in some cases as traumatizing; lack of possibility to find any interest in most of the mathematical issues presented – it seemed that dreariness was a built-in character of mathematics. Today, most of these subjects avoid any unnecessary contact with mathematics. They minimize their involvement with the mathematical studies of their children and if possible leave it entirely in the hands of their spouse. During the interview, when mathematical tasks were posed, they tended to look repelled and used expressions of anxiety ("this makes me sweat") along with claims of total "black-out" of some mathematical concepts or procedures.

Patterns of general opinions towards mathematics.

A. *Mathematics is an essential part of life.* This pattern includes the following ideas. Mathematics should be compulsory for all high school students. Anyone can reach a considerable level of achievement in this subject, if provided with guidance that meets the needs of the individual. Mathematics is an important language of communication. Mathematical knowledge is needed for successful functioning in society, and this includes not only arithmetic but also geometry, statistics, algebra and some basic concepts in calculus. An educated person is expected to have access to these topics.

B. *Mathematics is a powerful discipline.* This pattern emphasizes the importance of studying mathematics as a tool rather than acquiring knowledge. Dealing with mathematical material helps to develop a rational personality, and therefore it must not be cut off the compulsory curriculum in spite of understandable difficulties that can be overcome by persistence, patience and diligence. Accordingly, an educated person can be judged, among other things, by his or her ability to present clear and logical arguments. However, demonstrating familiarity with mathematical details is regarded as having less value than exhibiting knowledge in other cultural areas such as history, literature and art. It is presumed that for most professions the relevant mathematics is quite trivial..

C. *Mathematics is for the talented.* The principal claim here is that success in mathematics depends on innate capabilities. Dedication will not suffice unless "you've got it". Therefore, there is no point in compelling all students to study math beyond a certain basic level, usually seen as completed by the ninth or tenth grade. Up to this stage it is important to carry on in spite of possible dislike, for several reasons. First, arithmetic and simple geometry should be exercised, as they are useful in adult life. Second, acquaintance with this medium broadens the mind. Third, exposing everyone to mathematics is necessary for tracking the minority who will develop in this direction and eventually advance the prosperity of the whole community.

D. *Mathematics is an overestimated subject.* In this pattern the dominant theme is a sense of imbalance. The emphasis put on mathematics in high school is considered unjustified in light of the role it plays in post-graduate life for most adults. The priority given to mathematics is viewed as disproportional in regard to other important subjects. It is claimed that the high intellectual status attached to mathematics damages the confidence of students who prefer to concentrate on humanities and social sciences. Too often such students invest pointless - even if eventually successful - efforts in math studies, motivated solely by the current demand of universities to pass the matriculation exam in mathematics as prerequisite to entry. Turning math into an optional subject in high school is a therefore a desirable change. Mathematical knowledge is thus not valued as an essential part of what society might expect from the average educated person.

The patterns described above do not necessarily exclude one another. However, we found it feasible to ascribe one pattern to a subject, according to the most dominant theme in his or her response. The distribution of subjects within the types of mathematical self-schema and the patterns of opinions is presented in figure 1. Data include the subjects' professions, gender and level of math taken in high school.

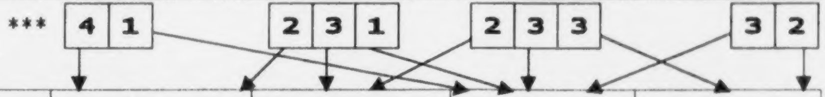
Concluding remarks

Several observations can be made in regard to figure 1. First, it is noticeable that of the 12 subjects who studied math at high or intermediate levels, 11 were identified with the positive or semi-positive self-schemas. All of the 12 subjects who studied math at the minimal level demonstrated the indifferent or negative schemas. This finding suggests that choices made by adolescents regarding the extent of their math studies are liable to effect their future habits as users of mathematics. Since it is also reasonable to assume that such choices are guided *a priori* by existing mathematical self-schemas, shaped before entering high school, it follows that major themes in these schemas are not likely to alter through time. Further investigations on this point might concentrate on the role of teachers in the process of forming self-schemas at early stages.

Second, it can be seen that there is no obvious relation between the subjects' schemas and their general opinions towards mathematics. In other words, a certain personal experience does not necessarily dictate a certain pattern of ideas about mathematics education. One example for this is the gynecologist, who had taken great pleasure in studying math but considers mathematics to be of low importance for most people and expects high schools to put emphasis on more relevant issues. An opposite example is that of the insurance agent, who experienced boring, routine math lessons, but still wishes math to be a compulsory subject for all students (In this case one might consider the need to appear as "socially correct" to have some effect).

In addition, a comment on gender should be made. Much has been said about mathematics and gender (see Fennema & Leder, 1993), and this paper was not intended to focus on this important issue. Nevertheless, we draw the reader's attention to the last two columns of figure 1: It can be seen that the indifferent self-schema is exhibited mostly by men (6 of 8), while the negative schema column consists of women only.

Type of mathematical self-schema	**Positive**		**Semi-positive**		**Indifferent**		**Negative**	
Attributes of subjects who demonstrated this schema: Profession, gender* and level of math taken in high school**	Lawyer	M,H	Businessman	M,H	Principal of a post-secondary institution	M,L	Museum director	F,I
	Gynecologist	M,H	Sound technician	M,H	Businessman	M,L	Teacher in adult school	F,L
	Psychologist	F,H	History teacher	F,H	Musician	M,L	Interior designer	F,L
	Physician	F,H	Lawyer	F,I	Insurance agent	M,L	Producer	F,L
	Banker	F,I	Architect	M,I	Accountant	M,L	Registered nurse	F,L
			Photographer	M,I	Government official	M,L		
					Art teacher	F,L		
					Midwife	F,L		
No. of subjects	5 subjects		6 subjects		8 subjects		5 subjects	



Pattern of general opinions towards mathematics	Mathematics is an essential part of life (6 subjects)	Mathematics is a powerful discipline (5 subjects)	Mathematics is for the talented (8 subjects)	Mathematics is overestimated (5 subjects)

Figure 1. Distribution of subjects within the 4 types of mathematical self-schema and the 4 patterns of general opinions towards mathematics (N=24).

Legend: * M= Male, F= Female. ** H= High level, I= Intermediate level, L= Low level.
*** $\boxed{\text{no.}}$ = Number of subjects within a schema holding a certain pattern of opinions as shown by the dart.

This state of affairs, found in a small but highly educated group of people, ought at least to provoke thoughts, if not beyond this.

Finally, we find it hard to ignore the following finding, shown in figure 1. Of the 24 subjects included in this research, 13 were identified with non-positive mathematical self-schemas, and 13 expressed opinions that basically agreed with Burke's claim, cited earlier in this paper. On one hand, this general look at the data is uncalled for, since it overlooks the more complex, heterogeneous nature of adult perspectives on high school mathematics, as we tried to communicate in this study. On the other hand, if more than half of these academically oriented participants relate to this issue in such a non-positive manner, one cannot help but wonder what is the situation within the wide population of high school graduates. This issue remains open.

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ACQUIRING THE CONCEPT OF DERIVATIVE: TEACHING AND LEARNING WITH MULTIPLE REPRESENTATIONS AND CAS

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Abstract

For the second time, two teachers taught an introductory program of differential calculus using an advanced calculator. An innovative derivative competency framework was devised, and proficiency in each representation determined using a set of twenty-one different differentiation competencies. Although each class achieved similar overall facility on the test items they exhibited different strengths. More students in Class A became proficient in using the graphical representation and in particular they were better at translating to a graphical representation. Class B developed more proficiency with the symbolic representation and again this was particularly evident in translating to the symbolic representation. These different results reflect different "privileging" of the teachers.

Introduction and conceptual framework

Recently, the advent of advanced calculators with graphical, numerical and symbolic computer algebra systems (CAS), has made calculus more accessible to a wider range of students, and provided impetus for research into how students acquire conceptual understanding of differentiation. Tall (1996) describes a set of component differentiation processes/concepts (in different representations) for derivative and Lagrange (1999a) discusses a set of "schemes of use" for functions and pre-calculus using CAS.

Kendal and Stacey (1999) report a 1998 teaching trial of introductory differential calculus designed to explore CAS use and assessment. The test items were mainly symbolic and each class made different use of CAS which related to different teacher "privileging". A follow-up teaching trial was conducted in 1999 with a new primary objective; to identify specific competencies acquired by students while learning differentiation for the first time using multiple representations of derivative with CAS available. This paper reports the conceptual understanding of differentiation acquired by each class using a set of test items specially designed to measure competence in numerical, graphical and symbolic representations.

A *derivative competency framework* was developed and consists of a set of twenty-one fundamental competencies specifically associated with differentiation. It has provided direction for data collection, analysis and interpretation. Each Test 1 item corresponds to one element of the framework, characterized by its *question type* and solution pathway involving a *process* (formulation, interpretation, translation, or combinations of them) in each *representation* (numerical, graphical and symbolic). (See page 3 for more detail.)

A second objective of both studies relates to teacher "privileging". Wertsch (1990) describes how different forms of mental functioning dominate in particular contexts and are influenced by a range of socio-cultural factors while Thomas, Tyrrell, and Bullock (1996) discuss the importance of a variety of teacher-related factors including attitude towards technology, and personal beliefs about the ways students should be taught. The first study showed that teacher "privileging" impacted on the ways students used the technology, and what they learnt. Teacher A "privileged" technology, symbolic algebra, and procedures for standard tasks; Teacher B, conceptual understanding, and by-hand algebra; and Teacher C, conceptual understanding and graphical methods. These privileging patterns resulted in different cognitive experiences and learning outcomes. Class A used computer algebra more frequently, and was more successful with symbolic items (less successful with conceptual items). While Classes B and C were more successful with conceptual items, Class B was more successful with by-hand algebra and Class C used graphical (non-calculus) methods more frequently as an alternative to symbolic procedures.

The purpose of this paper is to report some of the outcomes of the follow-up research project namely, the facility of each class with numerical, graphical and symbolic representations of derivative using test items based on the *derivative competency framework*. An attempt is also made to relate differences between the two classes to differences in the personal "privileging" of each teacher.

The teaching trial, research methods and data collection

In the repeat trial, Teachers A and B, at the same school (same identity as 1998) taught a slightly modified calculus program. (Teacher C from the 1998 trial did not participate). Both teachers helped revise the twenty-lesson introductory calculus program which was given a stronger emphasis on the concept of derivative in numerical, graphical and symbolic representations, and links between them, and a reduced emphasis on CAS. They subsequently taught it to Classes A and B, shared ideas about lessons, used a common teaching program and lesson notes they helped prepare, and gave students identical worksheets. Both teachers were experienced teaching with the TI-92 calculator. Prior to the trial, thirty-three Year 11 students (aged approximately 17 years) learnt how to use the TI-92 which they used optionally during the teaching program for all class work, homework and tests. The first author observed and audiotaped all the lessons, maintained a journal, interviewed the teachers individually before and after the program, and conducted task-based interviews with fifteen students. In contrast to the 1998 study where the ability levels of students in the three classes were normally distributed and evenly matched, the two classes had different distributions. School based assessments over two years (verified by a pre-test) showed that while both classes had a majority of weak students, Class A's higher average attainment (including algebra and graphs) was due to the presence of more highly competent students. Students completed questionnaires, challenging assignment questions and completed two written tests. (See sample items in Table 1.)

The derivative competency framework

The *derivative competency framework* consists of a comprehensive (minimum) set of twenty-one competencies associated with the concept of differentiation and a broad view of its structure is evident in Table 3. Each Test 1 item matches one specific competency, characterized by its *question type* and the two aspects of the solution pathway, *process* and *representation.*

- *Question type*; numerical, (n), graphical (g), or symbolic (s)
- *Process* involved; formulation, interpretation, formulation and interpretation, translation and formulation, or translation of derivative
- *Representation*; numerical (N), graphical (G), or symbolic (S)

The *question type* is associated with the words of the question. "Find a rate of change" is classified as numerical (n); "find a gradient", graphical (g); and "find the derivative", symbolic (s).

P*rocesses* include; formulation (the cognitive ability to recognize that a derivative is required and know how to find it), interpretation (the ability to reason about a derivative in natural language), and translation (knowledge that a derivative determined in one representation, has meaning in a different representation).

R*epresentation* is governed by the method associated with finding the derivative. An approximate numerical (N) derivative at a point, is determined as a difference quotient (its limit found by investigation); a graphical (G) derivative is determined as the gradient of the tangent to the curve at a point; and a symbolic (S) derivative is determined as a function by directly manipulating formulae or as the limit of a function. These methods are described by Dick (1996).

Individual competencies and Test 1 items

Only Test 1 is reported in this paper. It was used to identify which differentiation competencies each student had mastered and consists of 21 items, (7 in each representation). Each item is designed to match one competency of the *derivative competency framework.* Five individual competencies are embodied in the five sample Test 1 questions displayed in Table 1 and discussed below.

Q.1 "Find the derivative $f'(5)$" is a symbolic (s) question which requires a numerical (N) formulation (decision to find an approximate derivative using a difference quotient). The competency is *(s to N) translation* & *(N) formulation.*

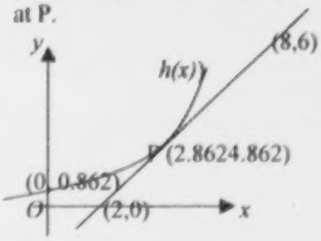
Q.2 "Give the reason" about a rate of change is a numerical (n) question, and requires reasoning about a numerical (N) derivative. The competency is *(N) interpretation.*

Q.3 "Find derivative" is a symbolic (s) question which requires a graphical (G) formulation (decision to find gradient of tangent at P). The competency is *(s to G) translation & (G) formulation.*

Q.4 "Find the gradient" is a graphical (g) question which requires a graphical (G) formulation (decision to find gradient of the tangent to the curve at $x = 3$). The competency is *(G) formulation.*

Q.5 "Find the rate of change" is a numerical (n) question which requires a symbolic (S) formulation (decision to find derivative using symbolic rule). The competency is *(n to S) translation* & (S) *formulation.*

Table 1. Sample Test 1 questions in each representation

	Representation	
Numerical (N)	**Graphical (G)**	**Symbolic (S)**
Q.1 The co-ordinates of points on a curve close to P where x = 5 are displayed in the table below. (Table is given). Find the best possible estimate of the derivative f '(5). **Q.2** At 1.00pm, the rate of change of temperature in your house is +3 degrees Celsius (°C) per hour. Immediately after 1.00pm, is the temperature in the house most likely to: decrease, stay the same or increase. Give a reason for your answer.	**Q.3** The graph of the function h(x) is sketched below. The tangent at point P, on the curve h(x) has also been drawn. Find the value of the derivative of h(x) at P.  **Q.4** Use a graph of $y = x^2 + x - 10$ to find the gradient of the curve at x = 3.	**Q.5** The height of a plant can be determined by the formula $H(t) = 7t^3 - 3t^2$, where H is the height of the tree in metres, and t is the number of years since the tree was first planted. Find the rate of increase of the tree's height, 2 years after it was planted.

Results

Class facility

Competency is demonstrated by the student if the written solution on the test item indicates awareness of all of the necessary cognitive steps. The competency is not demonstrated if *conceptual errors* are made, such as selecting an inappropriate representation, incorrect formulation (such as failure to choose two points on the tangent to the curve) or interpretation mistakes. However, competency may be demonstrated if *procedural errors* are made (such as algebraic, graphical, calculation, or careless mistakes e.g. transcription). Students from both classes used the calculator for numerical calculations, determination of a graphical derivative on Q.4 above, and only rarely for computer algebra.

On the twenty-one Test 1 items, the average number of competencies achieved by Class A (N = 14) students was 9.3 (St.D. 3.8), and Class B (N = 19), 9.7 (St.D. 3.4). *Class facility* is defined as the percentage of attempts where the specified competency is demonstrated so Class A's facility is 44% and Class B's 46%. The fairly low facility of each class was not entirely unexpected as each class has a majority of lower ability students (indicated by prior school testing). However, the almost identical facility was unexpected since Class A had a higher proportion of highly competent students (discussed earlier).

Class facility in each representation

Class facility on the 7 items in each representation, is determined by a variety of factors including *attempt rate* (percentage of attempts made to solve the specified items), the percentage of *valid* attempts (where an appropriate representation was used), and percentage *success* (where competency was demonstrated on valid attempts). A student is considered to be *proficient* in the representation if he/she demonstrates competency on at least four of the seven items in each representation. Table 2 displays these two characteristics, ***percentage of proficient students*** and factors which determined overall ***class facility*** in each representation.

Table 2. Percentage of proficient students and factors which determine class facility in each class and representation

		Representation		
Factors influencing class facility	**Class**	**Numerical (N)** (7 items)	**Graphical (G)** (7 items)	**Symbolic (S)** (7 items)
% of proficient students	**A** (N = 14)	29	50	43
	B (N = 19)	32	32	53
Attempt rate	**A** (N = 14)	84	87	83
	B (N = 19)	75	83	90
% of *valid* attempts (the representation used was appropriate)	**A** (N = 14)	78	80	81
	B (N = 19)	80	77	93
% *success* (competency was demonstrated on valid attempts)	**A** (N = 14)	54	68	70
	B (N = 19)	64	69	65
Class facility	**A** (N = 14)	36	48	47
	B (N = 19)	39	44	55

Symbolic representation: Class B has a higher proportion of symbolically proficient students (53%) and its superior symbolic facility (55%) is due to its higher attempt rate (90%), and on the (93%) of valid attempts, the representation was appropriate, affording an opportunity for success.

Graphical representation: Class A has a much higher proportion of graphically proficient students (50%) and its slightly superior graphical facility (48%) is due to its slightly higher attempt rate (87%) and marginally higher proportion of valid attempts (80%).

Numerical representation: The proportion of numerically proficient students is low in both classes (~30%). Class B's marginally better numerical facility (39%) is due to a higher success rate on valid attempts (64%) although Class A has a higher attempt rate (84%).

Class A's highest facility (with 50% of proficient students) is with the graphical representation (48%). This is almost identical its symbolic facility (47%) while its numerical facility is only 29%. In contrast, the highest facility for Class B (with 53% of proficient students) is the symbolic representation (55%) while its graphical (44%) and symbolic (32%) facilities are much lower.

Class facility in each representation sub-group

Each representation may be divided into three sub-groups, determined by the *question type* (described above). Table 3 shows the facility of each class (calculated as before) in each sub-group of competencies in each representation.

Table 3. Class facility for each representation sub-group on Test 1 items

			Representation	
	Class	**Numerical** **(N)** (7 items)	**Graphical** **(G)** (7 items)	**Symbolic** **(S)** (7 items)
Numerical ***question type*** **(n)**		**nN** (3 items)	**nG** (2 items)	**nS** (2 items)
	A (N = 14)	48	14	50
	B (N = 19)	63	3	55
Graphical ***question type*** **(g)**		**gN** (2 items)	**gG** (3 items)	**gS** (2 items)
	A (N = 14)	36	60	43
	B (N = 19)	26	65	63
Symbolic ***question type*** **(s)**		**sN** (2 items)	**sG** (2 items)	**sS** (3 items)
	A (N = 14)	25	64	48
	B (N = 19)	16	55	49

Compared to Class B, Class A's slightly superior graphical facility (48%, Table 2) is due to better facility with two of three graphical sub-groups, sG (64%) and nG (14%) and which involve translation to the graphical representation.

Class B's superior symbolic facility (55%, Table 2) relative to Class A is largely due to its much better gS (63%) and marginally better nS (55%) facility both of which involve translation to the symbolic representation.

Class B's marginally better numerical facility (39%, Table 2) is due to its better nN (63%) facility. Class A has better gN (36%) and sN (25%) facility, both of which involve translation to the numerical representation.

Class performance, teacher influences and CAS

The first author carefully observed every lesson and witnessed the different teaching practices of each teacher and learning outcomes of each class. Both teachers used CAS to demonstrate the planned conceptual activities (such as dynamic demonstrations to show the relationship between gradient of secant, curve and tangent to curve). Teacher A stressed routine procedures for routine numerical, and graphical tasks, emphasized algebraic routines and use of computer algebra, often used the calculator in front of the class, and permitted the students free use of the calculator. Class A developed equal facility with both graphical and symbolic representation, and compared to Class B, a higher proportion of graphically proficient students (Table 2) and better computer algebra capabilities (evident on Test 2) but of no benefit on the conceptual Test 1. Teacher A taught Class A students to respond to specific data cues and to use the representation suggested by the context of the question which often resulted in a translation to the graphical or numerical representation. As a consequence of Teacher A's emphasis on standard procedures and use of technology, Class A developed better facility with competencies involving translation to the graphical (and to the numerical) representation (Table 3) and in consequence a slightly superior graphical facility.

Teacher B taught for conceptual understanding and encouraged his students to develop their intuitive ideas about rate of change, slope and the limit concept. This contributed to Class B's superior facility with the nN sub-group (Table 3) which involves interpretation and formulation competencies, and a consequent slightly superior facility with the numerical representation. Teacher B consistently and deliberately stressed the importance of the symbolic derivative and Class B developed a higher proficiency with the symbolic representation than with the other two representations. In addition, compared to Class A, Class B had higher proportion of symbolically proficient students, attempted symbolic items more frequently, and made more valid selections (Table 2). Teacher B linked the numerical and graphical derivatives to the symbolic derivative by illustrating symbolic ideas with graphs and enactive representations (such as physical arm movements) to illustrate slopes of tangent lines. Class B's superior facility with symbolic representation, in particular competencies that involved translation to the symbolic representation, is a direct consequence of Teacher B's focus on teaching for conceptual understanding and use of the symbolic representation (including by-hand algebra). He allowed the students to use the calculator to determine graphical and numerical derivatives and for inducing the symbolic rules but otherwise insisted that students use by-hand algebra.

Discussion

Class A and Class B achieved an almost identical average number of differentiation competencies on Test 1 yet their facility in each representation and sub-group were different. Teachers A and B intended to teach the same material in the same way, however they taught with different personal interpretations and emphases. Both teachers' privileging patterns were identical to those in the first study. Once again, Teacher A "privileged" procedures for standard tasks, and use of technology including computer algebra, and Teacher B, conceptual understanding, the symbolic representation and by-hand algebra. In consequence, Class A and Class B students had different cognitive experiences and acquired different differentiation competencies which related directly to the ways they were taught. Class A students used technology more frequently, relied on standard procedures and developed both symbolic and graphical facilities to the same extent. Compared to Class B, Class A's slightly superior proficiency with the graphical representation particularly translation to the graphical (and to the numerical) representation, developed through greater use of the more intuitive, and visual graphical representation of derivative and use of context cues in the questions. In contrast, Class B students preferred and were more competent with the symbolic representation, particularly translation to the symbolic representation, significant in light of their weaker algebraic background and lower school performances. In an interview, Teacher B stated that he strongly directed his students towards the symbolic representation because of his personal beliefs that the symbolic representation was the most important, and that by-hand algebra was crucial for understanding. He was also convinced that his less able cohort of students would not cope successfully with more than one representation.

How important is by-hand practice for these essentially conceptual items? Lagrange (1999b) suggests that "techniques" play an important role in conceptual understanding. Techniques involve both the conceptualization of the steps required and their execution. Class B students used CAS for learning about the numerical, graphical and symbolic derivatives but not for executing algebra, yet they achieved higher facility on symbolic items. Although CAS use was optional, it was not needed or used on the test as the algebraic demands of items were simple. Is Class B's greater symbolic facility (in spite of poorer performance on school tests including algebra) due in part to Teacher B's insistence on by-hand practice?

During these lessons, the two classes achieved similarly overall but there were significant differences, which classroom observations linked with their teacher's "privileging". Is it possible that in the future, teachers using CAS will be able to identify their own "privileging" and modify their teaching strategies so that all students in their classes will have the opportunity to acquire the complete set of numerical, graphical and symbolic competencies associated with the concept of derivative? Was twenty lessons sufficient time for students to develop all competencies and does student aptitude and preference for representation place a natural limit on what can be achieved?

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YEAR 6 STUDENTS' PROGRESSION ON THE CORE MATHEMATICS COURSE OF AN INTEGRATED LEARNING SYSTEM

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This paper explores seven Year 6 students' progress on the core mathematics course of an integrated learning system (ILS) where individual students are presented with electronic worksheets in random form. It shows their frustration's, confusion and stress with the inflexible marking systems of the ILS, the repetitive formats of the ILS's worksheets and the unusual nature of setting-out procedures in some worksheets. It presents a model that relates student, teacher, classroom and operational characteristics to performance and progression on the ILS.

One of the by-products of the growth of information technology in education has been the computer-based integrated learning system (ILS). These systems have *substantial course content*, *aggregated record systems*, and a *management system* which "will update student records, interpret learner responses to the task in hand and provide performance feedback to the learner and teacher" (Underwood, Cavendish, Dowling, Fogelman, & Lawson, 1996, p. 33). They can be marketed to schools as the answer to students' numeracy difficulties and teachers' reporting requirements.

This paper reports on a study in which students used the core mathematics course of an ILS. The course is a closed system, that is, the curriculum content and the learning sequences are not designed to be changed or added to by either the tutor or the learner (Underwood et al., 1996). It is endorsed by the manufacturers only as a tool to consolidate already introduced material and to diagnose student difficulties with this material. It is divided into a range of *strands* (eg. numeration, addition, fractions, and volume) which are then sub-divided into collections of tasks that are sequenced in terms of performance at different *levels*.

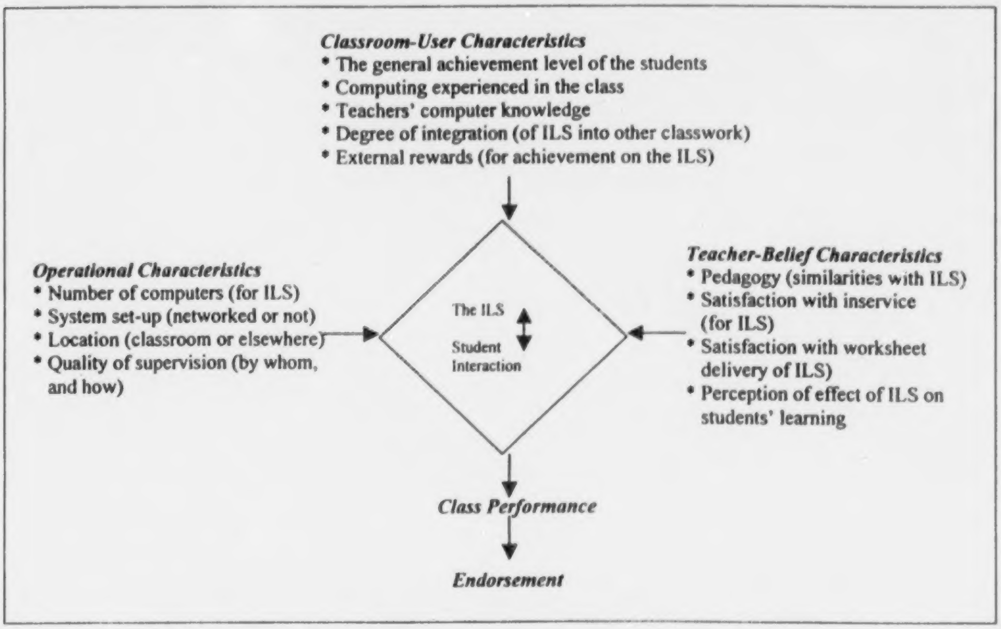
The mathematics tasks are in the form of individual, timed, randomly presented electronic worksheets. This allowed the ILS to place the students at their mastery level (70%) and to automatically raise them to the next level when they achieved high mastery (85% at one level was the same as 60% mastery at the next). The tasks are generally attractive in their presentation and sometimes creative in the way they probe understanding. They include 2-D representations of appropriate teaching materials (eg. Multi-base Arithmetic Blocks, Place Value Charts, fraction and decimal diagrams) and refer to online student resources (eg., help, tutorial, toolbox, audio and mathematics reference). However, some tasks had novel presentations and solution formats (eg. some algorithms required the ones to be typed first), other tasks had strict syntax and setting-out requirements (0.63 correct .63 incorrect), and accessing help and tutorial automatically graded performance as incorrect.

The management of the core mathematics course of the ILS does not appear to reflect effective mathematics teaching and computer use. Students are passive and do not construct knowledge or investigate (Wiburg, 1995), their initiative and autonomy are removed and they do not receive activities in sequence or work in groups (Sivin-Kachala, Bialo, & Langford, 1997). There is a tendency for tasks to be closed and based on speed and to have insufficient variety to prevent repetition.

There is very little evidence that the core mathematics course improves student learning (Becker, 1992), except for some topics in secondary school mathematics (Underwood et al., 1996). There is some evidence that it is possible to progress in the ILS without knowledge (Baturo, Cooper & McRobbie, 1999a) in a similar manner to how the Individually Prescribed Instructional (IPI) packages of the 70s allowed progression without facilitating mathematics learning (Erlwanger, 1975).

Context of the study

This study is part of the second stage of a three-stage evaluation of the ILS. The first stage focused on Years 1 to 10 classes as cases, identified factors that influenced teacher endorsement (see Baturo, Cooper, Kidman & McRobbie, submitted), and measured student achievement in relation to progress on the ILS (see McRobbie, Baturo, Kidman & Cooper, in preparation). Figure 1 provides a summary of the findings with regard to endorsement and illustrates the way factors effected performance on the ILS.



Classroom-User Characteristics
• The general achievement level of the students
• Computing experienced in the class
• Teachers' computer knowledge
• Degree of integration (of ILS into other classwork)
• External rewards (for achievement on the ILS)
Operational Characteristics
• Number of computers (for ILS)
• System set-up (networked or not)
• Location (classroom or elsewhere)
• Quality of supervision (by whom, and how)
The ILS
Student Interaction
Teacher-Belief Characteristics
• Pedagogy (similarities with ILS)
• Satisfaction with inservice (for ILS)
• Satisfaction with worksheet delivery of ILS)
• Perception of effect of ILS on students' learning
Class Performance
Endorsement

Figure 1. Factors that influence the endorsement of the ILS (modified from Baturo, Cooper & McRobbie, 1999b).

The second stage focused on 14 individual Years 6 and 8 students who had never worked on the ILS. The ILS was restricted to the topics of common and decimal fractions, multiplication, division and area to allow in-depth diagnostic interviews.

Stage 3 repeated Stage 2 but with a Year 6 and a Year 8 class that had been identified by the ILS manufacturers as the most successful users of their product in the Brisbane metropolitan area and with no restrictions on mathematics topics.

Method

This paper reports on the seven Year 6 students' acquisition of knowledge from the core mathematics course of the ILS via persistent detailed observations and regular individual diagnostic interviews.

Subjects. The seven Year 6 students were selected from 16 volunteers in a Year 6/7 composite class in a small Brisbane inner-city middle-class school. The school and teachers volunteered for the project and neither the teachers nor the students had used the ILS before. The seven students were selected on the basis of their responses to a diagnostic test covering representational, procedural and structural knowledge of whole numbers and fractions (Baturo, 1997). Two students were low achieving, two medium, two high and one very high. The students rotated through three 15-minute ILS sessions per week. The two teachers used a team-teaching approach with the composite class, and their management of learning appeared to be child-centred and constructivist. Because of the small number of students and the mathematics restrictions, there was no integration of the ILS with normal classroom mathematics.

Instruments. Data was gathered from six sources: (a) the DAN tests; (b) a pre- and post-test focusing on common and decimal fractions, multiplication, division and area; (c) a pre- and post-questionnaire covering affects and beliefs with respect to numeracy and computers; (d) two semi-structured interviews (one at the end of the placement period and one at the end of the trial); (e) observations of every ILS session; and (f) the ILS reports. This paper focuses on findings from the diagnostic tests, the interviews and the ILS reports. The first interview focused on how the students became familiar with the ILS environment; the final interview focused on the students' understanding of the concepts and processes covered in the ILS sessions, and their feelings about learning in this environment. ILS reports provided records of each student's time on the computer and gains in years.

Procedure. A laboratory of 2 computers was established in a small alcove at the back of the classroom and the ILS was programmed to provide only those tasks that related to common and decimal fractions, multiplication, division and area. It was not possible to isolate the computers, and users, from the remainder of the class. As a result, the ILS users were aware of all classroom occurrences.

Each student was enrolled in the ILS course at level 4.00. It was decided to enrol the students below their school year level as students in Stage 1 of the larger study had experienced early difficulties with the core mathematics course. The students were administered the pre-test and the pre-questionnaire and rostered for 3 ILS sessions per week during school time, ensuring that the times did not clash with classroom lessons that students particularly enjoyed (eg. art, sport, play time on the class's school computer). Each session was observed and videotaped, field notes were taken of interesting activity, and ILS reports printed. After the ILS had found their starting level, students were withdrawn from the classroom and given the first interview. At the end of the trial, students were given the second interview, the post-test and the post-questionnaire. The interviews were audiotaped and notes were taken on a specially prepared interview response sheet.

Analysis. The videotapes were viewed and instances of students' dilemmas as they got to know the ILS procedures were noted. These formed the basis of the items of the first interview, resulting in idiosyncratic first interviews contingent on the observed video behaviours. The observations and the interview results were translated into protocols and behaviours categorised, the questionnaire results were coded and these results combined with the test results and ILS reports from the management system to provide a profile of each student.

Results

The results for the diagnostic test and progress on the ILS (averaged across fraction, decimals, multiplication, division and area) are presented in Table 1. The highest test score (from a possible 185) was 177, while the lowest was 38. The results for the tests were supported by comments about the students' abilities by their class teachers. Each student's progress on the ILS was managed by the ILS management system, which interpreted the student's response to each exercise, and provided performance feedback. The feedback was accessed at the conclusion of every 3rd session.

Table 1.
Students' DAN test scores and progress on the ILS

	Diagnostic test results			Progress on the ILS					
Student	No.	%	Rating	3	6	9	12	15	18
Alan	177	95.7	VH	5.68	6.77	6.65	6.71	6.88	6.92
Ricky	144	77.8	H	4.17	5.24	5.60	5.65	5.72	5.81
Lance	133	71.9	H	5.06	5.35	5.53	5.58	5.62	5.77
Tim	95	51.4	M	4.32	4.49	4.09	4.11	4.24	4.38
Kara	103	55.7	M	4.12	4.98	5.13	5.22	5.35	5.47
Tracey	56	30.3	L	4.08	4.68	4.48	4.50	4.55	4.64
John	38	20.5	L	3.54	4.23	4.06	3.80	3.91	4.06

Note: Rating = VH - very high, H - high, M - medium, and L - low achieving.

Overall, although the classroom did have 2 other computers in it, none of the students equated mathematics with work on computers. Most of the students said they only used the computers for playing games. Hence, using computers for mathematics was a novel activity for the students, and it was made more novel by some of the unique presentation and solution formats in the ILS. Each student faced it differently.

Alan became very frustrated with the ILS. He knew the answers but found he could be marked incorrect because of typing errors (of which he made many) or differences with syntax. He became annoyed when a correct answer (eg. .63) was assessed as incorrect, but a different form of the answer (0.63) was correct. As a result of these errors, he received the same exercise 3 times in succession on a number of occasions, which further frustrated him. He was particularly frustrated that his competence with long division was not recognised. The ILS had forced setting-out procedures that were different to the classroom procedures Alan was accustomed to. He was continually assessed as incorrect when he did not exactly follow the protocols. Lance had the same experience with long division as Alan. Moreover, he found it stressful to be on the ILS and "missing out" on regular classwork. He was also enrolled in the ILS literacy package. When given a choice of the literacy or numeracy package, he invariably elected to do the literacy package. Tim was also enrolled in the ILS literacy package. He disliked having to do both packages on the same day, even though his sessions were staggered. Initially he was cooperative, but after 6 sessions he became bored and disruptive. He would allow the ILS to 'time-out' (exceed allowable time). When Tim realised an exercise was a repeat, he simply used the enter key until a new exercise was displayed. He complained the ILS was giving him "stuff that was too easy". He would ignore some exercises entirely. He said, "it was the font size's fault, 'cause it keeps changing". Apparently Tim becomes frustrated when required to read small print, hence he didn't attempt exercises which had a small font size.

Ricky was both stressed and confused by his interactions with the computer. On a number of occasions, the screen seized during a task Ricky could do, and all keys became inoperable. The ILS would then 'time-out' (exceed allowable question time) and Ricky would be marked incorrect. His frustration levels increased even further when the ILS delivered a full explanation on how to do the task. At other times, he was marked incorrect for not following a procedure exactly or for giving the answer in the wrong form and again the ILS would give an unnecessary explanation. This was particularly damaging because it confused Ricky as to what was mathematically correct. It did not seem to occur to Ricky that the ILS may have a fault; he thought there was a problem with his mathematics. Kara became frustrated in the division strand where an exercise was presented and required long division setting out, but short division was all that was required, or the answer could be determined mentally. Kara was presented with an exercise requiring her to "write an equivalent decimal to eight and forty-seven hundredths", but the term 'equivalent' was not in the ILS glossary, nor was the term 'equal'.

Tracey's reading difficulties created problems with the constantly changing font size. She said she didn't like the small writing because "it's for good readers". Tracey found several aspects of the ILS confusing. She didn't cope with the forced setting-out methods that were meaningless to her. It appears a graphics limitation of the ILS also confused Tracey. An analogue clock displayed 'quarter past 1' and requested

Tracey add 15 minutes. The answer of 1:30 was correctly given. After her third attempt at giving the same response, the ILS took over the exercise explaining how to add 15 minutes to 2:15 – the stimulus of the clock face was incorrect, yet Tracey was penalised. Tracey was confused by this and was convinced she must have misunderstood how to read the clock face or add the time. She did not realise it was an error in the ILS which was causing the confusion. John was confused by some of the inflexible solution formats. For example, in one task, he was required to find a missing fraction on a number line. He knew the answer to be one half but was marked incorrect when he typed in typed in ½, 0.5, and .5. When the ILS explained how to do the task, stated that he had to type "[1] [enter] [2]". John also had difficulties with a task that required him to work out the area of a 5x3 rectangular array. John measured this rectangle with a ruler from the ILS toolbox and found it was 7cm x 3.5cm. He became confused when he found he was supposed to use 5 and 3. He said he "was checking the sides to make sure they really were 5 and 3". He said "3 is OK, close enough, but the 5 and 7 can't be right". He then dismissed the exercise as "it was being dumb again".

By 3 months, a majority of the seven students disliked the ILS so severely that the trial had to be terminated early. The students made little progress after the 6th session with the lower-ability students being particularly erratic (see Table 1). The reason for this was the students' frustration, confusion and stress with the ILS. They did not enjoy being in the study; when it was their roster time, they would cringe and try to make excuses as to why they couldn't do their ILS session for that day. The net result was a negative attitude towards the ILS. This was highly evident for Tim and John who would not even attend the "thank you" party (lunch from a well known fast food outlet), they said they wanted nothing more to do with the ILS.

Discussion and conclusions

The programming limitations of the ILS mean that there is inflexibility in solution formats and repetitiveness in worksheet formats. These limitations can cause frustration, confusion and stress. The students were frustrated when there was too much repetition or when a correct answer was assessed as incorrect. In one situation, the ILS repeated the same exercise three times in succession; in another situation, a tutorial on how to get the answer had still to be sat through although the correct answers was given. The constantly changing font style and size was a problem depending upon the ability of the student. Brighter students dismissed this as unnecessary and "slack" on the manufacturer's part, but still considered the task. Less able students found font changes (especially small fonts) a deterrent. All students disliked having their typing expertise used as a determinant for their mathematics abilities.

The forced setting-out of the ILS, especially in the division and multiplication strands caused confusion. There was particular difficulty in recognising whether to include or omit the '0' ones in decimal fractions, whether to use decimal or common fraction notation (eg. $^1/_2$ or 0.5 etc.). Graphical inaccuracies also caused problems. As well, the overall experience of undertaking the ILS sessions was stressful for some students. Even though they were not on the ILS during favourite school subjects, they still disliked "missing out" on regular lessons. In there was choice, the literacy package was favoured. Technical glitches within the ILS causing the computers to seize were also a problem, as was the 'timing-out' aspect of the ILS.

The focus in this study (part of Stage 2) was on individual students. The model in Figure 1 was the result of a Stage 1 study on classes as cases. A new model (see Figure 2) is now necessary to encompass the detail required when analysis comes down to the student. However, Figure 1 still has application at the group level and shows that success depends on operational, classroom and teacher characteristics. In this trial of the ILS, there was no adequate supervision, little creative computing experience, little integration of the ILS with other class work, and no external rewards. As Baturo et al. (submitted) argued, these characteristics are associated with lack of success with the ILS. This is particularly so of supervision that has to assist the students deal with the novelty and the idiosyncrasies of the ILS. As well, the teachers' pedagogical approach was not really compatible with the ILS and they were not in a position to assist the students on the ILS. Therefore, it is not that surprising that the ILS was unsuccessful for the seven students and that they became dissatisfied with it.

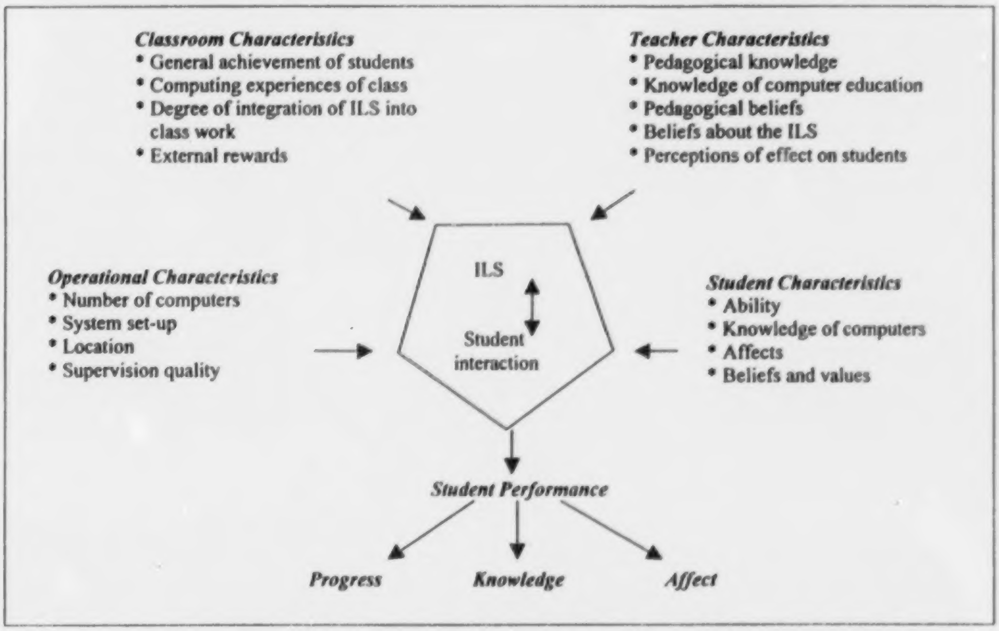

Classroom Characteristics
• General achievement of students
• Computing experiences of class
• Degree of integration of ILS into class work
• External rewards
Teacher Characteristics
• Pedagogical knowledge
• Knowledge of computer education
• Pedagogical beliefs
• Beliefs about the ILS
• Perceptions of effect on students
Operational Characteristics
• Number of computers
• System set-up
• Location
• Supervision quality
ILS
Student interaction
Student Characteristics
• Ability
• Knowledge of computers
• Affects
• Beliefs and values
Student Performance
Progress
Knowledge
Affect

Figure 2. Factors influencing students' performance on the ILS

However, in this study, how this dissatisfaction became apparent depended on the particular student. The more able students were able to understand the deficiencies of the ILS and, if motivated, would have been able to come to terms with it. The more committed students were willing to persevere and, given time, could have overcome their frustration's. As actor-network theory (Gaskell & Hepburn, 1998) describes, the effect of software like the ILS depends on the actors within the class – its actions shapes these actors and, in turn, are shaped by these actors. The ILS can not exist in isolation; it requires the interactions of a network of other actors as it is taken-up, modified, used and/or ignored (Gaskell & Hepburn, 1998). The ILS and the network evolve together with the result being a 'coursenetwork' (Gaskell & Hepburn, 1998). Thus, when the focus of the analysis is the individual students, as it is in this paper, there must be a focus on actors. Figure 2 illustrates the beginning of such an actor-based model. As is described in Baturo, Cooper, Kidman and McRobbie (1999b), this model begins to explain the actions of individual students with the ILS.

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SOLVING MULTIPLICATIVE WORD PROBLEMS WITH DECIMAL FRACTIONS: THE EFFECTS OF PROPORTIONAL REASONING AND METACOGNITION

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The purpose of this paper is to investigate the effects of proportional reasoning and metacognition on students' ability to solve multiplicative word problems with decimal fractions. In this study, 344 Japanese elementary school students in Grades 4, 5, and 6 were given a test involving multiplication problems, proportional reasoning problems, and metacognitive questionnaires. The study identified both proportional reasoning and metacognition as factors in the solving of multiplicative word problems. In particular, both proportional reasoning and metacognition were important factors in the Grade 4 students' work, whereas proportional reasoning was a more important factor than metacognition for the students in Grades 5 and 6.

A considerable body of research exists on the solving of word problems. It is widely known that many students have difficulty solving word problems. In particular, previous research has indicated that students especially have difficulty solving multiplicative word problems with decimal fractions (e.g. Fischbein *et al.*, 1985; Greer, 1994; Greer and Mangan, 1986). Fischbein *et al.* (1985) suggest a theory to account for the difficulty students have in solving multiplicative word problems. They suggest that "*each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model*" (p.4), which mediates how one chooses the operation needed to solve the problems. The primitive model associated with multiplication is "repeated addition." Because the primitive model obeys constraint that the multiplier must be a whole number, students would have difficulty solving word problems in which the multiplier is a decimal fraction.

To give the correct solution for a multiplicative word problem, a student needs not only to change his or her conceptual model of multiplication, but also to develop many other abilities, such as problem solving strategies, computation skills, proportional reasoning, metacognition, and others. This paper focuses on proportional reasoning and metacognition as significant factors in the solving of multiplicative word problems with decimal fractions.

Proportional reasoning refers to the ability to infer the value of one quantity if another quantity is changed, given that a proportional relationship exists between the two quantities (cf. Tourniaire, 1986). Without assuming a proportional relationship between the two quantities involved in such a problem, it is not possible to find the correct solution.

According to Flavell (1976), metacognition "*refers to one's knowledge concerning one's own cognitive processes and products or anything related to them*" (p. 232). In the process of solving multiplicative word problems, students use metacognition to confirm whether the result obtained by computation can be valid for the situation involved the problem.

The purpose of this paper is to investigate the effects of proportional reasoning and metacognition on students' ability to solve multiplicative word problems with decimal fractions.

METHOD

Subjects

The subjects who participated in the investigation were 313 pupils from 3 different elementary schools in Japan. The pool of subjects included 89 pupils in Grade 4 (9 or 10 years old), 101 pupils in Grade 5 (10 or 11 years old), and 123 pupils in Grade 6 (11 or 12 years old). In the Japanese curriculum, which all of the pupils had followed, multiplying with whole numbers is introduced in Grade 2 and multiplying with decimal fractions in Grade 5.

Instrument

A test used in the investigation was consisted of 4 multiplication problems, 4 proportional reasoning problems, and 12 metacognitive questionnaires. Table 1 shows the problems included on the test. These problems had been modified in an attempt to use terminology and notation more familiar to the Japanese population.

Table 1. Test Problems

Multiplication Problems

(1)1 liter of oil costs 600 yen. What is the cost of 0.3 liters?

(2)1m of iron pipe weighs 1.2 kg. How much does 0.8 m of iron pipe weigh?

(3)1 kg of oranges costs 580 yen. What is the cost of 2.4 kg?

(4)There are 1.2 kg of sauce per 1 liter. One restaurant uses 7.6 liters of sauce a month. How many kgs of sauce does the restaurant use per month?

Proportional Reasoning Problems

(1)The student is shown a subscription card from a popular magazine. It offers three plans: 1) A 6-month subscription for 3 payments of 4000 yen each; 2) A 9-month subscription for 3 payments of 6000 yen each; 3) A 12-month subscription for 3 payments of 8000 yen each. Do you get a better deal if you buy the magazine for a longer period of time?

(2)The student is shown pictures of two egg cartons, one containing a dozen eggs (8 white eggs and 4 brown eggs) and the other containing 1 1/2 dozen eggs (10 white eggs and 8 brown eggs). Which carton contains more brown eggs?

(3)The student is shown a picture of 7 girls with 3 pizzas and 3 boys with 1 pizza. Who gets more pizza, the girls or the boys?

(4)The student is shown a picture of two trees. Tree A is 8 feet high and tree B is 10 feet high. This picture was taken 5 years ago. Today, tree A is 14 feet high and tree B is 16 feet high. Over the last five years, which tree's height has increased more?

Metacognitive Questionnaire

A. Before you began to solve the problem- what did you do?
  (1)I read the problem more than once.
  (2)I thought to myself, do I understand what the problem is asking me?
  (3)I tried to remember if I had worked a problem like this before.

B. As you worked the problem- what did you do?
  (1)I thought about all the steps as I worked the problem.
  (2)I checked my work step by step as I worked the problem.
  (3)I did something wrong and had to re-do my step(s).

C. After you finished working the problem- what did you do?
  (1)I checked to see if my calculations were correct.
  (2)I looked back at the problem to see if my answer made sense.
  (3)I thought about a different way to solve the problem.

D. Did you use any of these ways of working?
  (1)I drew a picture to help me understand the problem.
  (2)I "guessed and checked."
  (3)I wrote down important information.

The multiplication problems were drawn from Kishimoto (1999). Within the set of multiplication problems, 4 addition and subtraction problems were also included so that students could not assume that multiplication was always the correct operation. The proportional reasoning problems were adapted from the work of Lamon (1993). The metacognitive questionnaires were adapted from parts of the questionnaires that Fortunato *et al.* (1991) used to assess metacognition in mathematical problem solving. The qualitative method of measuring students' metacognition using questionnaires is broadly accepted in the field of psychology (e.g., Okamoto and Kitao, 1992; Swanson, 1990). Subjects were asked to rate each of 12 items along a five-point scale ranging from 0 (not at all) to 4 (very well). The mean of the 12 values given by a student was regarded as the student's metacognitive ability.

Procedure

The test was administered to each of the subjects late in November 1999. The test took approximately 1 hour. The analysis of subjects' responses is organized by the subjects' grade in school.

RESULTS

Scores of correct responses

Table 2 shows the mean number of correct responses to the multiplication problems, proportional reasoning problems, and metacognitive questions in each grade. The mean number of correct responses to the multiplication problems was 0.82in Grade 4, 2.36 in Grade 5, and 3.39 in Grade 6. These values showed significant differences among all of the grades at the 1% level. The mean number of correct responses to the proportional reasoning problems was 1.12 in Grade 4, 2.07 in Grade 5, and 2.32 in Grade 6. These values showed significant differences between Grades 4 and 5 and between Grades 4 and 6 at the 1% level, as well as between Grades 5 and 6 at the 5% level. The mean number of correct responses to the metacognitive questions was 2.10 in Grade 4, 2.57 in Grade 5, and 2.67 in Grade 6. These values showed significant differences between Grades 4 and 5 and between Grades 4 and 6 at the 1% level. The mean number of correct responses to all types of problems used in the test-multiplication problems, proportional reasoning problems, and metacognition questions-increased as the grade level rose.

Table 2. Mean Number of Correct Responses in Each Grade

	Grade 4 (n=89)	Grade 5 (n=101)	Grade 6 (n=123)
Multiplication	0.82	2.36	3.39
Proportional reasoning	1.12	2.07	2.32
Metacognition	2.10	2.57	2.67

Correlation with multiplication

Table 3 shows the correlation coefficient of proportional reasoning and metacognition with respect to multiplication in each grade. The correlation between multiplication and proportional reasoning is high for all grades. The correlation coefficient for proportional reasoning combined across all grades is approximately 6.8. The correlation between multiplication and metacognition is high in Grade 4 (0.579), but it is low in Grade 5 (0.325) and Grade 6 (0.392). There were significant differences in the correlation coefficients of all items for all grades at the 1% level.

Table 3. Correlation with Multiplication in Each Grade

	Grade 4 (n=89)	Grade 5 (n=101)	Grade 6 (n=123)
Proportional reasoning	0.658	0.699	0.687
Metacognition	0.579	0.325	0.392

Contribution for multiplication

Table 4 shows the results of a multiple regression analysis and the corresponding statistical test for each grade. The coefficient of multiple determination in each grade is approximately 0.52. In other words, about 52% of the correct responses to the multiplication problems can be explained by the factors of proportional reasoning and metacognition.

The regression formulas in terms of multiplication for each grade are as follows:

Grade 4: Multi=0.38Prop+0.36Meta-0.37

Grade 5: Multi=0.65Prop+0.27Meta+0.32

Grade 6: Multi=0.57Prop+0.35Meta+1.13

Key to abbreviations: Multi: multiplication; Prop: proportional reasoning; Meta: metacognition

In Grade 4, the coefficient (0.38) for proportional reasoning is almost equal to that for metacognition (0.36). Proportional reasoning and metacognition almost equally affect the ability to solve multiplicative word problems in subjects in Grade 4. In Grade 5, the coefficient (0.65) for proportional reasoning is higher than that (0.27) for metacognition. In Grade 6, the coefficient (0.57) for proportional reasoning is also higher than that (0.35) for metacognition. We can conclude that proportional reasoning has a greater effect than metacognition on the ability to solve multiplicative word problems in subjects in Grades 5 and 6.

Table 4. Results of Multiple Regression Analysis

Grade 4

(1)Statistical Significance

Source	Degree of Freedom	Sum of Squares	Mean Square	F Value
Model	2	34.11	17.06	58.65**
Error	86	25.01	0.29	
Total	88	59.12		

**p<.01

(2)Partial Regression Coefficient

Variable	Parameter Estimate	Standard Error	t Value
Prop	0.38	0.05	7.01**
Meta	0.36	0.07	5.41**
Interception	-0.37	0.14	-2.56*

*p<.05, **p<.01

(3)Coefficient of Multiple Determination: $R^2=0.577$

Grade 5

(1)Statistical Significance

Source	DF	SS	MS	F
Model	2	67.17	33.59	53.09**
Error	98	62.00	0.63	
Total	100	129.17		

**p<.01

(2)Partial Regression Coefficient

Variable	PE	SE	t
Prop	0.65	0.07	9.20**
Meta	0.27	0.11	2.51*
Interception	0.32	0.29	1.11

*p<.05, **p<.01

(3)Coefficient of Multiple Determination: $R^2=0.520$

Grade 6

(1)Statistical Significance

Source	DF	SS	MS	F
Model	2	52.01	26.00	68.94**
Error	120	45.26	0.38	
Total	122	97.27		

**p<.01

(2)Partial Regression Coefficient

Variable	PE	SE	t
Prop	0.57	0.06	9.91**
Meta	0.35	0.09	4.03**
Interception	1.13	0.25	4.55**

**p<.01

(3)Coefficient of Multiple Determination: $R^2=0.535$

DISCUSSION

Change of student's cognitive structure

According to the regression formulas, both proportional reasoning and metacognition were found to be significant factors in the performance of multiplication by Grade 4 students, while proportional reasoning was found to be more significant than metacognition in the performance of multiplication by students in Grades 5 and 6.

This result shows a change over time in the students' cognitive structure. In Japan, the topic of rate is taught in Grade 5, and proportion is taught in Grade 6. The teaching of these two topics helps to promote the development of multiplicative conception in children. Inhelder and Piaget (1955) found that children's proportional conception would develop around age 11 or 12. For these reasons, students in Grade 5 or 6 have better developed proportional reasoning skills to call upon than do Grade 4 students as they solve multiplicative word problems. Grade 4 students must rely more exclusively on metacognition.

Contribution of proportional reasoning and metacognition to solving multiplicative word problems

The correlation between successful multiplication and proportional reasoning was high for students in all grades. This result shows that proportional reasoning is one of the initial factors needed for solving multiplicative word problems. Our results support the findings of Vergnaud (1983), who also asserted that solving multiplicative word problems requires the conception of ratio and proportion. Metacognition makes a slight contribution to the solving of multiplicative word problems, perhaps because it is a general factor that enhances problem solving in various domains.

Other factors in the solving of multiplicative word problems

Our model showed that proportional reasoning and metacognition alone account for a student's ability to perform multiplication in only 52% of cases. Both proportional reasoning and metacognition seem to be initial factors that allow students to solve multiplicative word problems with decimal fractions, but these two factors alone cannot fully explain the means by which children solve multiplicative word problems. Calculation skill and problem-solving strategy, among other factors, may also contribute. In order to best help students to solve multiplicative word problems with decimal fractions, we need not only to teach multiplication itself but also to assist students in developing other skills such as proportional reasoning and problem solving ability.

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DOES A RESEARCH BASED TEACHER DEVELOPMENT PROGRAM AFFECT TEACHERS' LESSON PLANS?

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This paper examines the effects of a teachers' in-service program on lesson plans written by the teachers as an indication to possible changes in their teaching. The program focused on students' misconceptions, possible incorrect responses and their sources and presented theories and research findings concerning students' ways of thinking. After participation in the course, teachers knowledge of students' mistakes and their sources improved. Lesson plans revealed that students' common incorrect responses (e.g. "multiplication makes bigger") were considered. It is argued that beliefs are an important factor that influences changes in teachers' lesson plans.

Promoting teachers' knowledge of students' thinking is a major goal of an increasing number of professional development programs for teachers of mathematics (e.g., Even, Tirosh and Markovits, 1996; Fennema, Carpenter, Franke, Levi, Jacobs, and Empson, 1996; Wood, 1998). Such programs attempt to help teachers understand the mathematical thought processes of their students. Only a few studies, however, explore the impact of these programs on what the teachers do in their classroom and on their students' learning. The 4-year longitudinal Cognitively Guided Instruction (CGI) teacher development project is one of the few professional development programs that has provided strong evidence that knowledge of students' ways of thinking can dramatically modify teachers' instructional methodologies. The observed changes in teachers' practices are attributed to two critical reasons: (a) The participants learned the specific research-based model that formed the basis of the teacher development program, and (b) they received ongoing support while using this model in their classrooms. Fennema et al. (1996) emphasize the importance of both components in their teacher development program. However, due to limited resources, many professional development programs consist of workshops with no (or extremely limited) support for teachers in their classrooms. Two questions that come to mind are: How does participation in a workshop that focuses on children's conceptions and misconceptions related to specific mathematics topics (with no support system in classrooms) influence inservice teachers' knowledge of children's thinking about this topic? Do teachers take their knowledge of what children know and understand into account when designing and teaching specific topics?

Previous studies have shown that participation in workshops focusing on children's perceptions of rational numbers can increase inservice teachers' awareness of students' incorrect responses and their possible sources (e.g., Klein, Barkai, Tirosh and Tsamir, 1998). The study describes the effect of a research based workshop that was specifically designed for enhancing inservice teachers' knowledge of students' ways of thinking about rational numbers on participants' design of lesson plans on multiplication and division word problems involving rational numbers.

Methodology

Subjects: Fourteen in-service elementary school teachers from the Tel Aviv area participated in the study. All except one were experienced teachers who taught mathematics in the fifth and sixth grades (where rational numbers are taught); one teacher taught mathematics in the lower grades. They all had at least five years of teaching experience. Teachers chose to participate in the course voluntarily and received official credit from the Ministry of Education. Ten of the teachers graduated from teacher education colleges; four had university degrees.

The Workshop titled "Students' ways of Thinking About Rational numbers" (STAR) consisted of 56 hours (14 meetings of 4 hours each). It aimed at enhancing teachers' knowledge of children's ways of thinking about rational numbers. Use was made of relevant research findings on children's conceptions of rational numbers (e.g., comparing fractions– Kidron and Vinner, 1983; Smith, 1995; comparing decimals- Nesher and Pleled, 1986; incorrect algorithms for calculating addition, subtraction, multiplication and division expressions- Ashlock, 1994; common, incorrect responses to multiplication and division word problems- Bell, 1982; Hart, 1981). Theories related to students' ways of thinking about rational numbers (e.g., Fischbein, Deri, Nello and Marino, 1985) and various, general teaching approaches for enhancing conceptual changes (e.g., The cognitive conflict teaching approach- Swan, 1983; The teaching by analogy approach- Clement, 1987) were described and discussed.

A substantial part of one of the meetings was devoted to a thorough discussion on lesson plans. Participants were presented with lesson plans written by teachers who were aware of students' mistakes and took them into account in their lesson plans and with lesson plans written by teachers who didn't consider students' ways of thinking in their planning. They were also presented with excerpts of class discussions of the teachers who wrote the lesson plans. The participants examined the extent to which teachers' awareness of students' ways of thinking was reflected in their lesson plans, discussed responses of these teachers to students' errors, and described their own possible reactions to these students.

Data Collection and Procedure: A Diagnostic Questionnaire (DQ) was administered to the participants during the first and the last meetings of the course to evaluate the changes in participants' Subject Matter Knowledge (SMK) of rational numbers and their knowledge of children's conceptions and misconceptions of rational numbers. The DQ related to comparison of and operations with rational numbers, multiplication and division word problems involving rational numbers and beliefs about multiplication and division with rational numbers. A typical item asked the teacher to solve the problem and list common incorrect responses and their possible sources. At the end of the course, participants responded to a modified version of the

DQ, which was individually tailored for each teacher. Each teacher received the items he/she answered incorrectly on the questionnaire given at the first meeting.

Participants were also asked, at the beginning (meeting 2) and at the end (meeting 13), to plan in groups a teaching unit either on "Multiplication and Division word problems with Rational Numbers (MDRN)" or on the topic they had been teaching in their classes at that time. They had to specify the main aims of the teaching unit, to relate to students' prior mathematical knowledge and to describe the teaching methods and manipulatives. They were then asked to prepare a lesson plan for one lesson on the chosen subject and to demonstrate how they would teach their students to solve specific, given word problems. Nine of the fourteen participants submitted a lesson plan on MDRN. At the end of the course, one of the nine participants who initially wrote a lesson plan on MDRN did not submit this final assignment. As a result, eight teachers submitted lesson plans on multiplication and division of rational numbers before and after participation in STAR. Since our main aim was to explore the extent to which the participants took students' ways of thinking into account in planning a teaching unit on MDRN, we describe in this paper the results of these eight teachers.

At the last meeting the participants were invited to a summative, individual interview with one of the researchers. During a typical interview the researcher discussed with the interviewees their performance on the diagnostic questionnaire, paying special attention to any item they answered incorrectly, probing possible reasons for their mistakes, and discussing possible ways to deal with students' incorrect responses. Their lesson plans were then discussed, with an emphasis on students' ways of thinking. The researcher presented some items describing students' hypothetical responses and interviewees were asked to relate to them.

All meetings were observed, audiotaped and transcribed. Teachers' worksheets were collected throughout the course and provided additional data on participants' SMK and on their Pedagogical Content Knowledge (PCK).

Results and Discussion

Before the course all teachers in this group provided correct answers to the computations (SMK "that") but had difficulties explaining why specific steps in the algorithms were taken (SMK "why"). For example, they could not explain why in division of fractions "we invert and multiply". Most participants were acquainted with students' incorrect responses to comparing fractions (PCK "that") but could not describe possible sources to such mistakes (PCK "why"). After STAR, SMK and PCK of all participants improved. The main improvement was observed on knowing "why": Most teachers could explain the various steps of the standard algorithms. They identified typical errors that students were likely to make and were aware of possible sources of these mistakes.

Participants developed their MDRN lesson plans in three groups: one with four teachers and two of two teachers each. We shall describe, for each group, teachers' lesson plans before and after STAR, emphasizing observed changes in their consideration of students' ways of thinking.

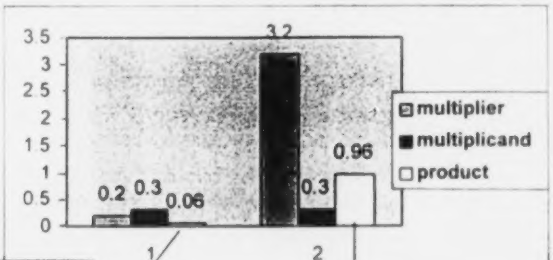
Group One (Aby, Rina, Jenny and Amy [1]):

Before participation in STAR, the four teachers in this group devoted most of the planning to a general discussion on the preferable type of an "opening task". Rina suggested to first present an authentic, complex situation, which could be deconstructed in various ways, demanding applications of various operations involving rational numbers. The others strongly opposed to this approach and suggested starting with a relatively simple, structured, standard problem, which could be solved using a mapping table, then gradually leading the students towards more complex problems, using mapping tables as well. When asked, specifically, what difficulties do they expect students to have with word problems involving rational numbers, they said they expected ***"no problems"***[2].

After STAR all teachers in this group decided to concentrate on structured activities that aimed at raising students' awareness that multiplication does not always make bigger and division sometimes makes bigger. Rina suggested a table of multiplication and division drills as an opening task, asking the students to compare the magnitude of the multiplier, multiplicand and the product. For example:

drill	product	conclusion
2x3	6	The product is bigger than both the multiplier and the multiplicand
¼ x8	2	The product is bigger than the multiplier and smaller than the multiplicand
0.2x0.3	0.06	The product is smaller than both the multiplier and the multiplicand

Aby suggested the use of an Excel graph for comparison. The Excel graph for the following expressions: 1) 0.2x0.3=0.06 2) 3.2x0.3=0.96



The product is smaller than both the multiplier and the multiplicand

The product is smaller than the multiplier and bigger than the multiplicand

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[1] All teacher names are fictitious.

[2] Quotes are presented in a different font

Jenny and Amy suggested starting their lesson plan with a word problem involving integers, the answer to which was a rational number, not expecting student difficulties ("*They* [students] *solved such problems in lower grades*"). Then, to present analogues word problems with proper fraction divisors, thereby raising students' awareness that "division sometimes makes bigger". They explained: "*We shall first deal with multiplication and division drills, to illustrate to students that multiplication does not always make bigger and division does not always make smaller*".

Teachers in this group believed that the use of mapping tables will guarantee that students will correctly solve multiplication and division problems. They said "*This procedure always works*" and "*You only have to learn the procedure, it becomes technical and automatic, and there are no problems*". Even after STAR the teachers in this group still argued that if teachers use algorithms and teach in a very structured way the students will know the material and make no mistakes. They believed that teaching needs to be subject matter oriented and structured. As a result they built the same activities for all students, addressing possible common students' errors and their sources. During the final interview they said "*After the course I look for reasons why the students gave a specific answer. Before the course I never thought about the influence of students' experience with integers on their performance with fractions. I didn't consider students' mistakes before the course*".

Group Two (Cindy and Ruth):

Cindy and Ruth referred to different types of division word problems (partitive division and measurement division) in their planning, both before and after STAR. They distinguished between "finding a part- when given the whole", and "finding the whole -when given a part" problems, as "*students find the first type easier*". They suggested sequencing word problems from easy to difficult ones. Cindy and Ruth were strongly in favor of using drawings as means for explanations. When the instructor explicitly asked them: "Have you related in your lesson plans (before STAR) to common incorrect students' responses?" They responded that they did not specifically refer to students' ways of thinking.

After STAR they differentiated between high achieving, intermediate and low achieving students and planned the lesson mostly for the intermediate group. Cindy and Ruth claimed that the high achievers would do well even without instruction and that no instruction could help low achievers. In their opinion, teaching has to be focused on the intermediate students, as they are the ones most likely to be influenced by teachers' actions. For these students they built tasks and sequenced them, as they did before STAR, from easy to difficult ones, starting with 'finding a part when given a whole", which is, in their opinion, easier for students. They were aware of students' misconceptions and said they would address students' problems as they arise. For example, they suggested to present 5 half apples and to ask, "*What do we have here?*" The teachers explicitly stated that "*Students will probably answer, one of the following answers: 5 half apples, 5 apples, 2 ½ apples or 5 halves*". They advised the teacher to write students' answers on the board and specified how to relate to each (correct or incorrect) answer. They also suggested building the rest of the lesson on students' answers. For example, when a student answers "2 ½ apples", the teacher could ask him to explain his answer. They expected the explanation $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$ and suggested that the teacher addresses multiplication as repeated addition $5\mathrm{x}\frac{1}{2}$. Interestingly, when asked how they would deal with students' incorrect answers, Cindy and Ruth resented writing incorrect responses on the board. They expressed a belief that some students might remember the incorrect responses and think they are correct (this was in conflict with their suggestion to write both correct and incorrect responses on the board).

In sum, Cindy and Ruth were aware of students' common mistakes. They focused on intermediate students and in order to deal with their mistakes they sequenced tasks by level of difficulty.

Group Three (Carol and Wendy):

Like the teachers in the second group Carol and Wendy sequenced activities from easy to difficult ones, before and after STAR. They made distinctions between "finding a part-- when given the whole", and "finding the whole --when given a part" and suggested that teachers work first on word problems involving "finding a part" since this type of task is easier for students. After the course they were acquainted with the intuitive models of multiplication and division, and were aware of errors students were likely to make, but didn't plan the lesson accordingly. They gave no suggestions as to how to deal with students' incorrect responses when they appear. For example, Carol and Wendy referred to students' incorrect responses to the word problem "*Mother had 60 shekels. She paid ¼ of the money for groceries. How much did she pay for groceries?*" They said *"They* [students] *will divide instead of multiply, because when they divide, the answer is smaller"*, but did not know how to deal with them. Carol and Wendy believed in teaching for understanding and in lots of drill and practice. They said, "*We will work on each step thoroughly, so the students will understand the meanings*".

Although Carol and Wendy were aware of students' mistakes no change concerning students' ways of thinking was observed in their lesson plans.

Final comments

In this paper we reported on the effect of a teachers' in-service program on participating teachers' lesson plans. A workshop was especially designed for enhancing inservice teachers' knowledge of students' ways of thinking about rational numbers. This workshop focused on students' misconceptions, possible incorrect responses and their sources and presented theories and research findings concerning students' ways of thinking. After participation in the course, Teachers' SMK and PCK improved. The main improvement was exhibited in teachers' knowledge "why".

What can be said about the impact of participation in a course that focuses on children's ways of thinking about non-negative rational numbers and their possible sources on in-service teachers' lesson plans? (and more specifically the extend to which they consider students' ways of thinking in their planning)

Before the course only a few written references to possible, common incorrect students' responses were made in lesson plans. We hypothesized that after STAR teachers would adjust their lesson plans, taking account of common, systematic students' conceptions and misconceptions when planning their instruction. The results show that all teachers were aware of students' common errors but the ways and the extend to which they considered them varied. Teachers in the first group believed in subject oriented instruction. Their lesson plans after STAR showed the biggest change. They built structured activities for all students in order to deal with possible common misconceptions, and were in favor of teaching algorithms. Teachers in the second and third groups believed in students driven instruction. They sequenced tasks by level of difficulty from easy to difficult, were aware of students' difficulties and suggested dealing with them only if and when they occur. However, whereas teachers in the second group suggested how to deal with students' errors and misconceptions when they arise, teachers in the third group gave no suggestion how to deal with students' mistakes..

All the teachers in this study participated in STAR and showed improvement in their SMK and PCK, yet STAR had a different impact on their lesson plans. Clearly, teachers beliefs about teaching are critical factors which determine their teaching (Lerman, 1999). Research shows that change in teachers' knowledge without change in teachers' beliefs is not significant (Fennema, Carpenter, Franke, Levi, Jacobs and Empson, 1996). Our data show that teachers' beliefs are an important factor that influence changes in teachers' lesson plans.

Our small sample included only three groups of teachers. We suggest that further studies with a larger numbers of participants be conducted to investigate the impact of inservice programs that focus on students' ways of thinking on teachers' lesson plans and on teachers' actual teaching practices.

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A RESEARCH ON THE VALIDITY AND EFFECTIVENESS OF "TWO-AXES PROCESS MODEL" OF UNDERSTANDING MATHEMATICS AT ELEMENTARY SCHOOL LEVEL

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Abstract: *The understanding mathematics in the process of teaching and learning school mathematics has been a main issue buckled down by some researchers in PME. Koyama (1992) discussed basic components that are substantially common to the process models, and presented the so-called "two-axes process model" of understanding mathematics as a useful and effective framework for mathematics teachers. This research focuses on examining the validity and effectiveness of this model. By analyzing data collected in the case study of third grade mathematics class in a national elementary school, the validity and effectiveness of the "two-axes process model" of understanding mathematics is exemplified. The teaching and learning of mathematics that enables children to understand mathematics deeply and in their meaningful way is characterized as a dialectic process of individual and social constructions. The important teacher's roles are also suggested.*

Introduction

The word "understanding" is very frequently used in both the descriptions of objectives of teaching mathematics in the Course of Study (Ministry of Education, 1989) and in the mathematics teaching practices in Japan. The putting emphasis on understanding mathematics should be desirable in mathematics education, but what it does mean is not clear. Moreover, It is an essential and critical problem for us to know what mathematics teachers should do in order to help children understand mathematics. However, it also has not been made clear sufficiently.

The key for the solution of these educational problems, in my opinion, is ultimately to capture what does it mean children understand mathematics and to make clear the mechanism which enables children's understanding of mathematics develop in the teaching and learning of mathematics. In other words, it might be said to "understand" understanding. It is, however, not easy and we need our great effort to do it. The problem of understanding mathematics has been a main issue buckled down by some researchers, especially from the cognitive psychological point of view in the international group for the psychology of mathematics education (PME). As a result of their works, various models of understanding as the frameworks for describing aspects or processes of children's understanding of mathematics are presented (Skemp, 1976, 1979, 1982; Byers and Herscovics, 1977; Davis, 1978; Herscovics and Bergeron, 1983, 1984, 1985, 1988; Pirie and Kieren, 1989a, 1989b; Hiebert and Carpenter, 1992).

Theoretical Background:
The "Two-Axes Process Model" of Understanding Mathematics

Koyama (1992) discussed basic components that are substantially common to the process models, and presented the so-called "two-axes process model" of understanding mathematics. This model of understanding mathematics consists of two axes. On the one hand, the *vertical axis* implies some hierarchical levels of understanding such as mathematical entities, their relations, and general relations, etc. On the other hand, the *horizontal axis* implies three learning stages of intuitive, reflective, and analytical at each level of understanding. Through these three stages, not necessarily linear, children's understanding is able to progress from a certain level to a next higher level in the process of teaching and learning mathematics.

Intuitive Stage: Children are provided opportunities for manipulating concrete objects, or operating mathematical concepts or relations acquired in a previous level. At this stage they do *intuitive thinking*.

Reflective Stage: Children are stimulated and encouraged to pay attention to their own manipulative or operative activities, to be aware of them and their consequences, and to represent them in terms of diagrams, figures or languages. At this stage they do *reflective thinking*.

Analytical Stage: Children elaborate their representations to be mathematical ones using mathematical terms, verify the consequences by means of other examples or cases, or analyze the relations among consequences in order to integrate them as a whole. At this stage they do *analytical thinking*.

There are two prominent characteristics in the "two-axes process model". First, it might be noted that the model reflects upon the complementarity of intuition and logical thinking, and that the role of reflective thinking in understanding mathematics is explicitly set up in the model. Second, the model could be a useful and effective one because it has both descriptive and prescriptive characteristics.

Purpose and Method

The "two-axes process model" of understanding mathematics is expected as a useful and effective framework for mathematics teachers. In order to demonstrate the usefulness and effectiveness of the model, it is a significant task for us to examine both validity and effectiveness of the model in terms of practices of the teaching and learning of mathematics. Koyama (1996) focused on the validity of the model and demonstrated the validity of three stages at a certain level of understanding mathematics by analyzing data collected in a case study of fifth grade elementary school mathematics class in Japan.

The purpose of this research is to examine the validity and effectiveness of this model more closely by analyzing data collected in the case study of third grade mathematics class at the national elementary school attached to Hiroshima University. This school has two classrooms at each grade from the first to the sixth. Children in a classroom are heterogeneous in the same way as a typical classroom organization in Japanese elementary schools, but their average mathematical ability is higher than that of other children in the local and public schools. The mathematics teacher involved in this research is a collaborating member of our collaborative research project on mathematics education at Hiroshima University. He is an experienced and highly motivated, and has a relatively deep understanding of both elementary school mathematics and children.

A Case Study of Elementary School Mathematics Class

The data was collected in the case study of third grade (9 years old) mathematics class. There were 37 children (20 boys and 17 girls) in the classroom and the mathematics teacher was Mr. Wakisaka. The researcher and teacher made the case study in the mathematics class for introducing fractions to third graders. It was our main objective of the class that children were aware of the possibility of representing fractional parts by an idea of division into equal parts through such activities as placing over and folding fractional parts.

Making a Plan based on the "Two-Axes Process Model"

Therefore, using the "two-axes process model" of understanding mathematics, we planned the class in due consideration of both teaching materials of fraction and the actual state of children in the classroom as follows.

Firstly, we decide to make children use a fictitious unit named *"gel"* as a restriction so that they may construct various ways of representing fractional parts. Secondly, we give children two different fractional parts being based on the fact that as a result of our prior investigation the presentation of not one but two different fractional parts is effective for children be aware of representing fractional parts by making a comparison between them. Moreover, we use 3/5 cup of juice and 2/5 cup of juice as two fractional parts and two rectangular figures of them drawn on a sheet of paper in order to make it possible for children to notice the sum of them or the difference between them, and to place over and fold them. Thirdly, we put great emphasis on making a connection of children's various ways in representing fractional parts. In a whole-classroom discussion, we ask children to report their own solution for representing fractional parts, and encourage them to examine and refine their solutions by discussing about the similarities and differences among them, and then to share the value of various ways of representing fractional parts with others in the classroom.

Putting the Plan into Practice

The process of teaching and learning in the classroom progressed actually as follows. In the following protocol of the class, sign Tn and sign Cn mean the nth teacher's utterance and the nth child's utterance respectively.

Firstly, in order to make an emphasis on the process of abstracting fractional parts, the teacher Mr. Wakisaka told a story using two different bottles of juice, poured each of them into four same-sized cups, and then introduced rectangular figures to represent the volume of them. The teacher planned to make children use a fictitious unit named *"gel"* as a measure of volume. Therefore, he began to tell a fantastic story to his children. [Setting a Problematic Situation]

T1: *Today, let's make a space travel!*

CC: *Yes! Let's go!*

T&CC: *Four, three, two, one; fire!* (For a while)

T2: *Now, we have arrived at a jellyfish-planet. Jellyfish-aliens welcome and give us two different bottles of juice.* (He showed two bottles of juice. He poured each of them into four same-sized cups, and then he introduced rectangular figures to represent them. He prepared 1 3/5 cup of blue-colored juice and 1 2/5 cup of red-colored juice. But children were given no detail information about their fractional parts.)

T3: *Let's inform jellyfish-aliens of the volume! You should pay attention to the fact that jellyfish-aliens use a unit named "gel" as a measure of volume. But they understand such numerals as 0, 1, 2, 3, 4,...*

CC: *Only gel?*

T4: *Yes! So jellyfish-aliens can't understand units of measure with which you are familiar.*

CC: *Do you mean we can't use units such as dl or ml?*

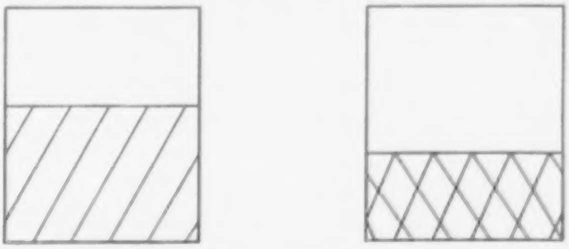
T5: *Yes, I do. You can't use those units on the earth.*

C1: *How should we do to inform them of the volume? If we could use such units as dl and ml, it would be easy. But it is really difficult.*

C2: *We had better to make another convenient unit named "bal."*

T6: *Inform them of the volume of juice somehow.*

In the next step, the teacher gave each child a sheet of paper on which two rectangular figures for two fractional parts of juice were drawn as shown in Figure 1. Children individually did manipulative or operative activities on their own way using the paper for a while. Some children measured the length, while some children cut two fractional parts out of a sheet of paper, placed one figure over another one, and then folded them. During children's activities, the teacher was observing it. [Intuitive Stage]



Blue-colored Juice (FPB)　　Red-colored Juice (FPR)

Figure 1. Rectangular figures for fractional parts

The following shows some types of solution for representing fractional parts of blue-colored juice (FPB) and red-colored juice (FPR) that children invented as a result of their activities at the intuitive stage.

(1) FPB is half and just a little more. FPR is just a little less than half.
(2) The length of FPB is 4*cm* 7*mm*, so it is 4*gel* and 7*gel*.
(3) We had better to make another unit named *deci-gel* as a smaller unit than *gel*.
(4) FPB is less than one *gel*, so it is zero *gel*.
(5) FPB is zero *gel* and 47 *deci-gels*.
(6) The sum of FPB and FPR is one *gel*.
(7) When I divide the whole length of a rectangular figure by the unit of 1*cm*, it can be divided into eight equal parts. FPR is equal to just three parts, so it is zero point three *gels*.
(8) When I place FPR over FPB and fold them by noticing the difference between them, FPB is equal to three parts and FPR is equal to two parts.
(9) When I divide the whole length of a rectangular figure into ten equal parts, FPB is equal to six parts and FPR is equal to four parts.

In the following step, the teacher asked children to report their own solution for representing fractional parts, and organized a whole-classroom discussion in order to encourage them to examine and refine their solutions. During the discussion, children have paid their attention to their own manipulative or operative activities and represented their own solution in terms of figures or daily-life and mathematical language already learned. [Reflective Stage]

T7: *Now, we inform jellyfish-aliens of the volume of fractional parts.*
C3: *I can say that FPB is half and just a little more and that FPR is just a little less than half.*
C4: *I have a question. I think the expression such as just a little more/less is not clear.*
T8: *Indeed, so it is. But, how should we do?*
C5: *May I use centimeter?*
T9: *How do you think?*
CC: *We can't use centimeter.*
T10: *Yes, you are right. We can use only gel as a unit. Is there another idea?*
C6: *I have a good idea. FPB is 4gel and 7gel. FPR is 3gel and 1gel.*
CC: *What does it mean?*
T11: *Anyone who can explain this idea?* (For a while)
C7: *Oh, I see. I'm sure that he measured the length with a rule.*
CC: *Yes, we agree with you. The length of FPB is 4cm 7mm, so he changed both units of cm and mm to gel.*
T12: *I see, so it is...*
C8: *Why do you use the same unit gel? They, cm and mm, are different units.*
C9: *Because we are allowed to use only gel as a unit!*
C10: *I say FPB is 4gel and 7deci-gel, because the units of cm and mm are different and we already learned other units of liter and deciliter for measure of volume.*
CC: *That is a good idea. It is easy to represent fractional parts with such a new unit of*

volume.

C11: *But, I think that the expression 4gel and 7deci-gel is not reasonable, because 4gel is more than 1gel.*

C12: *So, my idea is better than it. FPB is less than 1gel, so it is zero gel, because we know that the number smaller than 1 is 0.*

C13: *Your idea of zero gel is not reasonable, because it means that there is nothing.*

C14: *Then, zero gel and 47deci-gel is a more reasonable expression.*

These children's way of representing two fractional parts seemed to be mainly based on using an idea of length and inventing new units of volume. The teacher wanted to change the line of such discussion and encouraged children to present other ideas.

T13: *I see. Are there any other ideas?*

C15: *When I divided the whole length of a rectangular figure by the unit of 1cm, it was divided into eight equal parts and FPR was equal to just three parts. So FPR is zero point three gels.*

C16: *What does it mean?*

C17: *It means that FPR is less than one gel and that FPR is equal to three parts.*

C18: *I think that the expression such as FPR is zero point three gel is a similar expression as FPB is zero gel and 47deci-gel.*

C19: *Even when we divide the whole length into ten equal parts, could you still say FPR is zero point three gels? If FPR was equal to four parts in case of ten-parts division, would you express it as zero point four gels?*

T14: *You mean that we can say FPR is zero point four gels in such a case as it is equal to four parts.*

C20: *I did not consider such case.*

T15: *Do you understand what he wanted to do?*

C21: *It is similar to my idea, but I divided the whole length of a rectangular figure into five equal parts. It is a reason that when I placed FPR over FPB and folded them by noticing the difference between them, each of them could be divided into five equal parts. Then, FPB is equal to three parts and FPR is equal to two parts.*

Finally the teacher encouraged some children to report their own way of representing fractional parts by a division into equal parts, and aimed to help all children be aware of the possibility of representing fractional parts by an idea of division into equal parts. [Analytical Stage]

C22: *When I divided the whole length into ten parts, FPB is equal to six parts and FPR is equal to four parts.*

C23: *Mr., I think there are many ways of division into equal parts.*

CC: *Yes! There are many ways.*

T16: *Oh, there are many ways?*

C24: *Yes! There are as many as we like by multiplying the number of division, for example 8, 16, and 24 or 5, 10, and 15 etc.*

T17: *So, in the next class, we will continue to investigate the way of representing fractional parts less than one gel by using this idea of division into equal parts.*

Conclusion

When we made the plan of this mathematics class for introducing fractions to third graders, we used the "two-axes process model" as a framework and embodied it with teaching materials of fractions in due consideration both of the objectives of the class and the actual state of children in the classroom. In the classroom, the restriction of using a fictitious unit named "*gel*" imposed on children, the choice of 3/5 cup and 2/5 cup of juice as two fractional parts, and the two drawn rectangular figures with 8cm length given to children could make it possible to set the problematic situation where they individually invent a new unit and construct various ways of representing fractional parts less than one "*gel*" [intuitive stage]. As a result of their manipulative or operative activities with those figures followed by the constructive interaction in a whole-class discussion, children by themselves examined and refined their solutions [reflective stage]. Finally, children integrated some ideas for representing fractional parts, and in more general sense they could be aware of the possibility of representing fractional parts by using an idea of division into equal parts as a result of social constructions [analytical stage].

Through this research, we find out the followings. First, we can exemplify the validity and effectiveness of the "two-axes process model" of understanding mathematics by this case study of the elementary school mathematics class. Second, we might be able to characterize such a teaching and learning of mathematics that enables children to understand mathematics deeply and in their meaningful way as the dialectic process of children's individual and social constructions in their classroom.

In order to realize such mathematics classroom, it is suggested that a teacher should make a plan of teaching and learning mathematics in the light of "two-axes process model", and embody it with teaching materials of a topic in due consideration both of the objectives and the actual state of children. The teacher also should play a role as a facilitator for the dialectic process of individual and social constructions through a discussion with children and among them. There are two important features of teacher's role in the process of teaching and learning mathematics. The one is related to children's individual construction and it is to set the problematic situation where children are able to have their own learning tasks and encourage them to have various mathematical ideas and ways individually. The other is related to children's social construction and it is to encourage and allow them to explain, share and discuss their various mathematical ideas and ways socially in the classroom.

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THE EFFECTS OF DIFFERENT INSTRUCTIONAL METHODS ON THE ABILITY TO COMMUNICATE MATHEMATICAL REASONING

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ABSTRACT

The main purpose of the present study was to investigate the differential effects of metacognitive training and cooperative learning on the ability to communicate mathematical reasoning. Participants were 384 eighth-grade students who studied the "linear function graph" unit under four instructional methods: cooperative learning embedded within metacognitive training (COOP+META), individualized learning embedded within metacognitive training (IND+META), cooperative learning with no metacognitive training (COOP), and individualized learning with no metacognitive training (IND). Results showed that the COOP+META group significantly outperformed the IND+META group, who in turn significantly outperformed the COOP and IND groups on various aspects of verbal explanations on graph interpretation test.

Interest in communication is both more widespread and more central to mathematics education reform efforts than ever before. The NCTM (1989) reforms emphasize the importance of problem solving and communicating mathematical ideas, not simply isolated answers. Mathematical communication "requires attaining abilities to read, write, explain, discuss, justify, and clarify mathematical reasoning using different forms of representations" (Elliott & Kenney, 1996. p. ix.). Nevertheless, recognizing the centrality of communication as an issue for mathematics education is necessary but not sufficient to ensure a higher frequency of communication. Even when there is a high level of interest and commitment to communication as a feature of mathematics instruction, many teachers may struggle with the challenges arising from implementing these beliefs in classrooms (Silver & Smith, 1996). There is an important need to investigate different instructional methods that can contribute to the attainment of mathematical communication in the classrooms. Communication is the essence of the small-group experience. To foster the ability to communicate mathematical reasoning it is only natural to give students the opportunity to study in small groups where their interactions are enhanced. This is in contrast to "traditional instruction (which) places most students in a position of almost total dependence on the teacher. Students seem to learn by listening and watching the teacher do mathematics and then by trying to solve the problems on their own" (Frederiksen, 1994, p. 536) rather than by being involving in mutual reasoning and resolution of cognitive conflicts that arise during the interactions. Research has suggested, however, that for positive outcomes to occur, small-group activities must be structured to maximize the chances that students will engage in questioning, elaboration, explanation, and other verbalizations in which they can express their ideas and through which the group members can give and receive feedback (Slavin, 1989).

These recommendations have led researchers (King, 1994; Mevarech & Kramarski, 1997) to suggest the structuring of group interaction through metacognitive training that enhances students' understanding of the task, awareness and self-regulation of strategy application, and connections made between prior and new knowledge.

The method of Mevarech & Kramarski, (1997), called **IMPROVE** emphasizes the importance of mathematical communication throughout the entire curriculum by changing classroom organization into small groups, learning and providing each student with the opportunity to do mathematics by involving him or her in mathematical communication via the use of metacognitive questions that focus on: (a) the nature of the problem (b) the construction of relationships between previous and new knowledge solved in the past; and (c) the use of strategies appropriate for solving the problem.

The purpose of the present study is to investigate the differential effects of metacognitive training and cooperative learning on the ability to communicate mathematical reasoning.

The line of reasoning presented before led us to hypothesize that being exposed to cooperative learning embedded within metacognitive training (COOP+META) would facilitate the ability to communicate mathematical reasoning more than being exposed to individualized learning embedded within metacognitive training (IND+META) which in turn would facilitate mathematical communication more then cooperative (COOP) and individualized (IND) settings with no metacognitive training.

METHOD

Participants

Participants were 384 students (181 boys and 203 girls) who studied in twelve eighth-grade classrooms randomly selected from four junior high schools, three classes in each treatment. The schools were an integrated school composed of students from different socio-economic status as defined by the Israel Ministry of Education.

Measures

The ability to communicate mathematical reasoning was assessed by the Graph Interpretation test focusing on analyzing verbal explanations. A 36-item test, adapted from the studies of Mevarech and Kramarski (1993), assessed students' ability to interpret graphs, particularly linear graphs. The test involved 25 multiple-choice items and 11 short open-ended items regarding basic knowledge about the Cartesian System and linear-graph interpretation. The test involved items that required qualitative and quantitative graph interpretation skills. The short open-ended items asked students to give a final answer and to explain their reasoning in writing.

Scoring: For each item, students received a score of either 1 (correct answer) or 0 (incorrect answer), and a total score ranging from 0 to 36. Kuder Richardson reliability coefficient was .91.

Verbal explanations: Each item on the mathematical explanations was scored on three dimensions: Correctness, fluency in providing different kinds of correct explanations and mathematical representations.

Correctness: Explanations could be correct or incorrect, supported by different kinds of arguments, and formulated by formal or informal mathematical language. An explanation was considered as correct if the argument fit the conventions, even if it was not stated in a formal way. For example, if a student argued that the change-rate of line A is greater than the change- rate of line B because "line A is steeper than line B", that argument was considered correct even though it was not phrased with formal mathematical concepts such as slope.

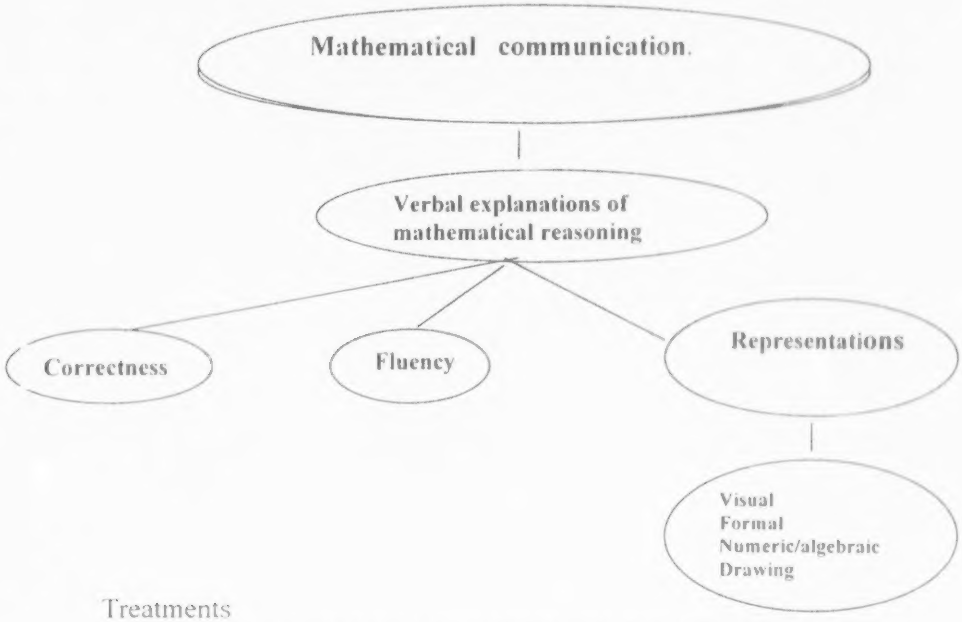
Scoring: For each item, students received a score of either 1 (correct explanation) or 0 (incorrect explanation), and a total score ranging from 0 to 11.

Fluency in providing different kinds of correct explanations: Students could use one or more arguments to explain their reasoning.

Scoring: The number of correct explanations a student provided for each item.

Mathematics representations: Students' mathematical explanations were classified into four categories: (a) verbal arguments based on visual analysis of the graph (e.g., line A is steeper, line A is more diagonal);
(b) verbal arguments based on formal concepts (e.g., the change-rate of line A is bigger because its slope is steeper than of line B);
(c) numerical/algebraic arguments (e.g., the change-rate of line A is three times more than the change rate of line B); and (d) arguments based on drawings that students added to the graph (e.g., adding one-unit steps to the graph and calculating the change rate by using the steps). Two judges who are expert in mathematics education analyzed students' explanations. Inter-judge reliability coefficient was .88.

Figure 1 summarizes the relationships addressed under mathematical communication in this study.


Mathematical communication.
Verbal explanations of mathematical reasoning
Correctness
Fluency
Representations
Visual
Formal
Numeric/algebraic
Drawing

Treatments

All classrooms studied the Linear Graph unit five times a week for two weeks. In particular, in all classrooms students studied the concept of slope, intersection point, and rate of change; (b) quantitative and qualitative methods of graph interpretation; and (c) transformation of algebraic expressions of the form y=mx+b into graphic representations. The metacognitive instruction used in the present study is based on the techniques suggested by Mevarech and Kramarski (1997) in the IMPROVE method. The metacognitive instruction utilizes a series of self-addressed metacognitive questions: comprehension questions, strategic questions, connection questions and reflection questions.

In addressing comprehension questions, students had to read the problem, describe the concepts in their own words, and try to understand what the concepts meant. The strategic questions are designed to prompt students to consider which strategies are appropriate for solving the given problem and for what reasons.

Connection questions prompt students to focus on similarities and differences between the graph at hand and graphs they had already interpreted or to compare different intervals on the same graph.

In doing so, students gradually learn to construct a network of information or a schema for understanding.

Reflection questions prompt students to focus on the solution process and to ask themselves "what am I doing here? "does it make sense?" "what if?" The metacognitive questions were printed in Students' Booklets, Teacher Guide, and on the hand held index cards that students used in problem solving. Students used the metacognitive questions orally in their small group/individualized activities, and writing when they used their booklets.

Learning instructions

COOP+META condition: Students in this condition studied in small heterogeneous groups composed of four students: one high achiever, one low achiever, and two middle achievers, using the metacognitive questions described above.

IND+META condition: In this condition, the metacognitive instruction was exactly the same as in the above condition, except that the metacognitive instruction was implemented in individualized rather then cooperative settings.

COOP condition: Under this condition, students studied in small heterogeneous groups as in the COOP+META condition, but they were not exposed to the metacognitive training.

IND condition: Under this condition students learned individually with no metacognitive training. This group served as a control group.

RESULTS

ANCOVA analysis was performed on graph interpretation achievements and on the various aspects of verbal explanations controlling for pretreatment differences.

Graph Interpretation: Table 1 indicated that although no significant differences were found between treatment groups prior to the beginning of the study, significant differences were found at the end of the study. Post-hoc analyses of the adjusted mean scores based on pair-wise technique indicated that the COOP+META group significantly outperformed the IND+META group who in turn significantly outperformed the COOP and IND groups, but no significant differences were found between the two groups who were not exposed to the metacognitive training.

Table 1: Mean scores, adjusted mean scores, and standard deviations on graph interpretation test by time and treatment

		COOP+META N=105	IND+META N=95	COOP N=91	IND N=93	F(3,380)
Pretest	M:	15.5	14.4	14.2	16.0	1.86
	S:	6.4	6.4	5.9	6.2	
Posttest	M:	24.4	20.9	19.2	19.8	18.44***
Adjusted	M:	24.0	21.4	19.8	19.1	
	S:	7.2	6.9	6.4	6.6	

***p<.001

Verbal explanations: Table 2 indicated that although no significant differences were found between groups prior to the beginning of the study on both correctness and fluency, significant differences were found at the end of study. Yet, post-hoc analyses based on the pair-wise technique indicated different patterns of performance on both measures. On correctness, the COOP+META group outperformed all other groups, but no significant differences were found between the IND+META, COOP, and IND groups. On fluency, the COOP+META group outperformed the IND+META group who in turn significantly outperformed the COOP and IND groups, but no significant differences were found on that measure between the two groups who were not exposed to the metacognitive training.

Table 2: Mean scores, adjusted mean scores, and standard deviations on verbal explanations by time and treatment

		COOP+META N=105	IND+META N=95	COOP N=91	IND N=93	F(3,380)
Correct Explanation						
Pretest	M:	2.9	2.7	2.9	3.1	0.32
	S:	2.5	2.3	2.2	2.6	
Posttest	M:	6.5	4.4	4.0	4.1	26.43***
Adjusted	M:	6.5	4.5	4.0	4.0	
	S:	3.1	3.1	2.7	2.6	
Fluency in Providing Different Kinds of Correct Explanations						
Pretest	M:	3.2	2.8	2.7	3.7	2.26
	S:	2.4	2.2	2.1	2.6	
Posttest	M:	8.9	6.5	4.9	4.6	22.55***
Adjusted	M:	8.9	6.7	5.2	4.2	
	S:	5.0	4.4	3.2	2.9	

***P<.001

Mathematical representations: Table 3 indicated that most students relied on numerical and algebraic representations in justifying their reasoning. Interestingly, the individualized groups (with or without metacognitive training) did so even more frequently then the cooperative groups (with or without metacognitive training). In addition to using numerical and algebraic representations, quite often students used verbal-formal representations. The frequency of using verbal-formal representations, however, was significantly larger under the COOP+META condition than under all other conditions. These differences were statistically significant (Chi square=27.0, $p<.0001$). Further analyses showed that under all conditions, students used the visual and graphic representations quite infrequently (less than 5% of the students).

Table 3: Frequencies (percent in parentheses) of students who used mathematical representations in justifying their correct mathematical reasoning by time and treatment.

	COOP+META N=105	IND+META N=95	COOP N=91	IND N=93
Visual explanations				
Pretest	6 (1.6)	3 (.8)	4 (1.0)	5 (1.3)
Posttest	7 (1.8)	7 (1.3)	5 (1.4)	0 (0.0)
Formal explanations				
Pretest	19 (4.9)	16 (4.7)	6 (1.6)	18 (4.7)
Posttest	31 (8.1)	11 (2.9)	13 (3.4)	19 (4.9)
Numeric/algebraic explanations				
Pretest	50 (13.0)	46 (12.0)	47 (12.2)	40 (10.4)
Posttest	54 (14.1)	61 (15.9)	55 (14.3)	60 (15.6)
Drawing				
Pretest	0	0	0	1 (.3)
Posttest	6 (1.6)	5 (1.3)	0	0
Total				
Pretest	75 (19.5)	65 (16.9)	57 (14.8)	64 (16.7)
Posttest	98 (25.5)	84 (21.8)	73 (19.0)	79 (20.6)

CONCLUSIONS

It was found that cooperative learning embedded with metacognitive training is effective in developing the ability to communicate mathematical reasoning in the classroom on three dimensions of verbal explanations: correctness, fluency and representations. The results indicate that verbal explanations improve understanding on graph interpretation. These findings support earlier conclusions. Cohen (1996) indicated that features of discourse are new behaviors that students can learn through practice and reinforcement. "Giving reasons for ideas", for example, can become a norm of behavior that enhances mathematical thinking and communication. Mevarech & Kramarski (1997) state that being presented with explanations related to why and how a certain solution to a problem has been reached, the student is given the opportunity to elaborate upon the information inherent in the explanations, and thus, learn from them. More theoretical conclusions and practical implications will be discussed on the presentation. In addition there will be presented more details on the metacognitive training and examples of students' verbal explanations regarding each instructional method.

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NARRATIVE ELEMENTS IN MATHEMATICAL ARGUMENTATIONS IN PRIMARY EDUCATION

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Abstract

Results from two research projects about processes of argumentation in primary mathematics classrooms will be presented. The central research interest is to examine the relationship between the participation of students in argumentative processes and their individual content-related development. Hereby, the focus is on mathematics teaching and learning situations in regular classroom settings. Illustrated by the interpretation of a solving process of third-graders this paper shows the narrative character of their argumentation-process. The theoretical relevance and some practical implications of this approach will be outlined, finally.

Introduction

Some results from two related research projects about processes of argumentation in primary mathematics classrooms will be presented. The projects are "Arguing in primary mathematics classrooms" sponsored by the state of Baden-Württemberg, Germany in 1994 and 1995, and "The reconstruction of 'formats of collective argumentation' in primary mathematics education" sponsored by the German Research Foundation (DFG) since October 1996.

The central research interest is to examine the relationship between the participation of students in interactional classroom processes and their individual content-related development. Hereby, the focus is on mathematics teaching and learning situations in regular elementary classes. The theoretical orientation of these projects is derived from ethnomethodology (GARFINKEL 1967), symbolic interactionism (BLUMER 1969) and cultural psychology (BRUNER 1996). From these perspectives classroom situations are seen as *processes of interaction*: Students and teachers contribute to their accomplishment according to their insight into the sense and purpose of these events. They act as it seems *sensible* and *tenable* to them. For this, they *interpret* these classroom situations: they reflect, set up and review hypotheses, and make rational decisions; common features are found which temporarily enable them to cooperate. In such a classroom situation they develop their content-related understanding in order to be mutually regarded as responsible and capable and in order to participate in the joint creation of this interaction. Thus, in such a situation their mind is challenged, which they employ and develop by the way.

In the project the research approach concentrates on (collective) argumentation (see KRUMMHEUER 1995; KRUMMHEUER & YACKEL 1990). This paper stresses the narrative character of this process. In the first chapter the view on learning as a social phenomenon will be outlined. Here, the concepts of "culture" and "classroom culture" play a crucial role. The second chapter deals with the narrative feature of those classroom cultures. In the third part an example of collective task solving processes among three third-graders will be presented in which the narrative character of their joint argumentation process will be explained. The last chapter summarizes the results, integrates them in the larger research-context and shows practical implication.

1 The social constitution of learning

In the discussion about conditions of learning it is more frequently stressed that the entire phenomenon of human learning is not spanned if one confines one's studies to the interior, mental processes of the learner. Learning is also a social process which takes place in the interaction between human beings:

> ... where human beings are concerned, learning ... is an interactive process in which people learn from each other (BRUNER 1996, p. 22; see also BAUERSFELD 1995, BRUNER 1990, ERICKSON 1982, and KRUMMHEUER 1992).

Thus, learning is socially constituted, or, in other words: interactive processes are vital parts of the "nature" of learning.

This insight led BRUNER 1986 to formulate the feature of a psychology which constitutionally considers social elements for psychic processes, as for example in the course of learning. He defines this as *cultural psychology* (p. 35) and explains that each individual development must be expressible in the particular symbolic system of a given culture. For this, the members of a culture not only have the general means of their language but, additionally, they can also employ specific culturally accomplished ways to interpret the psychological disposition of individuals. BRUNER 1990 defines this as *folk psychology* (e.g. p. 33ff). Regarding the teaching- learning- process in a classroom situation he speaks of *folk pedagogy* (BRUNER 1996, p. 46). These two concepts include the mostly implicit basic assumptions of a culture about the psychological functioning of its members.

> All cultures have as one of their powerful constitutive instruments a folk psychology, a set of more or less connected normative descriptions about how human beings 'tick', what our own and other minds are like, what one can expect situated action to be like, what are possible modes of life, how one commits oneself to them, and so on. We learn our culture's folk psychology early, learn it as we learn to use the very language we acquire and to conduct the interpersonal transactions required in communal life (BRUNER 1996, p. 35).

In this quotation it is also stated that such folk psychological insights are acquired at an early age and that they are already linked with the learning of one's own native language. The folk psychology is acquired through narrative interaction, in other words, their insights are learned from stories. BRUNER designates this way of "learning from stories" as an independent mood. On the one hand he contrasts this mood with a logical and scientific way of thinking, on the other hand, he regards it as a learning mechanism through which a child develops fundamental views and perspectives of the world it lives in (s. 1986, p. 11ff and 1996, p. 39f).

In this sense, learning is not only the appropriation of culture, it is implemented in its co-creation. Especially with regard to primary education it is often stated that basic cultural techniques such as reading, writing and arithmetic are taught and acquired, here. From the developed perspective of cultural psychology, this seems to be an insufficient point of view: Children do not only learn the contents of culture. Rather, through their contributions in reading, writing, and calculating they also create "a" or "the" culture. Concatenating these two aspects one arrives at what could be described as *classroom culture*. Participation in this double sense integrates the social constitution of learning. Thus, *classroom culture is a culture of subject matter and a culture of learning.*

2 The narrativity of classroom culture

Obviously, in primary education children like to listen to and to tell stories. The argument presented here goes beyond this empirical evidence and claims that the children learn by these stories. Hereby they learn the content of different school subjects. The classroom culture of our primary schools is characterized by narrativity: frequently, the different contents are presented in a narrative style, and the social constitution of classroom-learning can be described in related models of participation in situations of story-telling. This is also relevant for mathematics classes, and the analysis of processes of interaction concerning this subject matter can demonstrate the importance of this thesis in general.

In the following, firstly, some characteristics of such narrations in the observed mathematics classes will be presented. The aim is to clarify which aspects of narrativity are most relevant in mathematics classroom-interaction. It will be shown that it is of an argumentative nature and that argumentation dissolves in a narrative presentation (see also the concept of "reflexive argumentation" in KRUMMHEUER 1995). This means: The narrative classroom culture of primary education is based on rationality, and the social constitution on classroom-learning is the participation in the interactional accomplishment of argumentative, narratively structured sequences of actions.

This thesis does not imply that in classroom-situations "stories" will be told endlessly and that beyond educational goals in native language classes, classroom-education intends to teach children the telling of stories. Rather it is that the negotiated theme in a classroom-interaction emerges more frequently in such a way that one can reconstruct aspects of a narrative process. Thus the concept of "narration" is used here in order to describe a specific phenomenon of everyday classroom-conversation. It is not meant in the sense of literary science.

According to BRUNER 1990 one can identify four characteristics for narrative accomplishments: its

1. "sequentiality"
2. "factual indifference between the real and the imaginary"
3. "unique way of managing departures from the canonical", and
4. "dramatic quality" (p. 50).

Here, the first and third points are of special interest. The claimed narrativity of classroom-culture is seen in the patterned sequentiality of classroom-interaction. The specifity of an event, such as the elaborated solving process for a new mathematical task, is presented in relation to the canonical management of such events or problems.

In the following, more evidence will be given to this theoretical approach by reconstructing in detail a classroom-episode.

3 An example

This example might help to illustrate and differentiate the thesis: Two or three children work together in order to solve a given problem. A successful cooperation demands two different cognitive achievements from these children. They have to clarify

- what shall be done at what time and
- who shall do it at what time

Both cases have to do with the decision about the right moment whereas

- the first point deals with the insight within a sequence of solving steps and their chronological order and
- the second point has to do with the sequence of interactional moves and the chronological change of speakers.

The structure of actions and interactions with regard to the first issue can be called an "academic task structure" (ERICKSON 1982). It is based on the understanding of the situation of the problem as shared among the students. It is not identical with logical considerations about a sequence of solving steps according to the subject matter of mathematics (see VOLLMER & KRUMMHEUER 1997). A second issue can be described in reference to ERICKSON as a "social participation structure". Both structures are mutually dependent (see ERICKSON 1982, p. 156 and VOLLMER & KRUMMHEUER 1997).

In the following, I will illustrate some aspects of the academic task structure (ATS) by presenting an example of the mentioned research projects. This example will help to clarify four general aspects of the ATS.

The boys Daniel, Slawa and Stanislaw from a third grade are confronted with the presentation of numbers at the back of T-shirts which represent the first parts of a number sequence. Their task is to determine the fifth element of this sequence which is:

{3 - 8 - 15 - 24 - ?}

For this sequence, Slawa can quickly give a solution:

47 Slawa *(pointing at the picture)* Here comes five, here comes seven´..

49 Slawa here comes *(.)* nine´

52	Slawa	He gets an eleven-
53	Daniel	Why eleven´
54	Stanislaw	Why´
55	Slawa	Well eleven. Look´, *(precariously whispering)* how much plus three,
56		look´, at this number. five-
57<	Daniel	Well´, from three to eight are five.
58<	Slawa	*(directed to Daniel and still pointing at the picture)* here comes already
59		seven´, seven-
60	Daniel	seven-
61	Slawa	nine´ *(.)* eleven.
62	Stanislaw	*(inarticulate)* well yeah.
63	Slawa	Eleven plus twenty-four. add it here. then one gets *(figures about 2 sec)* thirty-five.

From a mathematical stance, one can view in Slawa's solution the thematization of the general concept of the sequence of differences and the first four numbers of a specific sequence of differences {5 - 7 - 9 - 11}. The boy cannot name them. He does not define them explicitly and in a certain way he is not talking about them, but through them. His two classmates cannot follow him. Slawa is obliged to explain; generally, he reacts in the way just described: He names the four elements of the sequence of difference. One short episode might demonstrate this:

77<	Slawa	This are five. here *(points at paper)* then seven´, here comes nine
78<	Daniel	five *(mumbles inarticulately)* from eight to fifteen are seven´
79	Slawa	add always two to it.
82<	Slawa	Thus here comes eleven´, Daniel *(points at number sequence)* here
83<	Daniel	seven´ yes from, yes
84		comes eleven to that number
85		from fifteen to twenty-four are nine.
86	Slawa	Thus here you get thirty-five . *(inarticulate)* thirty-five.
87	Stanislaw	Whoop-
88	Daniel	Yes, nine´

One recognizes how Daniel in <78, 84 and 88> agrees to the numbers 5, 7 and 9 as the difference between the given elements of the initial number sequence. He and Stanislaw as well do not conceptualize the numeration of these numbers as the elements of a number sequence which emerges by finding the differences. Even Slawa's meta-comment about the rule for this sequence of differences in <79> does not help. Slawa's finding of the solution, his presentation and his justifications are narratively oriented. In order to understand his solution one must, firstly, recognize the phenomenon of a sequence of difference and, secondly, the defining characterization $x_{n+1} = x_n + 2$ while repeating the numbers 5, 7, 9, 11. If somebody cannot infer this argument from the numeration of the numbers, he does not understand the sense of the story at all.

Summarizing, the four following conclusions can be drawn from the interpretation of this episode:

1. The mathematical concepts which are necessary for understanding the ATS are not introduced explicitly. In a narrative way, they are rather pointed at implicitly. Not all students are able to recognize the ATS by this way of presentation. The plausibility of this solving process might be inscrutable for them.
2. For the accomplishment of the different steps of the ATS the boys need certain mathematical competencies such as addition and subtraction of positive integers.
3. Only few or no meta-commentaries about the functionality of the ATS or certain steps of the ATS are given. This characterizes narratively organized interaction.
4. The presentation of the solution steps proceeds mainly by verbalization. The boys do not use alternative presentations such as visualization or embodiments. This is an additional characteristic for narratively organized processes of interaction in mathematical group work.

These four issues describe aspects of the ATS and its narrative generation in group work. In general these are "(a) the logic of subject matter sequencing; (b) the information content of the various sequential steps; (c) the 'meta-content' cues toward steps and strategies for completing the task; and (d) the physical materials through which tasks and tasks components are manifested and with which tasks are accomplished" (ERICKSON, 1982, p. 154).

4 Social learning conditions in classroom interaction and an implication for enhancing the classroom culture

The basic insight of this research is that in classroom teaching and in group work as well a proved folk psychology of learning in this classroom culture becomes apparent which for sure is not given by nature or God but which has two very important features: It functions in everyday classroom situations and it has a rationality.

The rationality of actions expresses itself in the pursuing or novel creation of a sequence of working steps. With regard to ERICKSON 1986 it is called the "academic task structure" (ATS). This is a sequence of actions as it is accomplished *by the participants* in their process of interactional negotiation. In primary mathematics classes, this interactive realization occurs often in a narrative style: the conducted calculations are told according to the sequence of the ATS in as much as the necessary competencies can be integrated. Typically, the inner logic of the total approach within such narratively realized academic task sequences is not explicitly thematized, but it is expected as a specific achievement of the participants. They have to *infer* this inner logic from the specific presentation of the narrations (for more details see KRUMMHEUER 1995, 1997). This does not usually happen successfully and often not in its entirety. Learning which is related to novel concepts and insights does not happen automatically. But, on the other hand, this kind of narrative classroom culture is characterized by a great stability in everyday primary school teaching and learning situations and there are many students who daily proceed successfully in their content-related learning development by participating in this classroom culture.

With regard to those students who do not proceed successfully, one issue will be emphasized, here: It is the fact that writing and application of other illustrating tools are missing. The observed interaction processes in the two projects are solely based on oral exchange. BRUNER 1996 speaks with regard to this point of the necessity of an externalization tenet.

> Externalization, in a word, rescues cognitive, activity from implicitness, making it more public, negotiable, and 'solidary'. At the same time, it makes it more accessible to subsequent reflection and meta-cognition (BRUNER 1996, p. 24 f).

The starting point of my argument is that in the project episodes preferentially verbal productions can be observed. Generally, the quick evaporation and the situative uniqueness of verbal accomplishments impedes the reflection on such interactive procedures - at least for some, the so-called "weak" students. Complementing such reflections with a written presentation of not only the result, but especially of the process of working seems helpful. BRUNER 1996 refers to the concept of the œuvre of the French psychologist Ignace Meyerson. Œuvre does not mean a somehow standardized scientific presentation. It rather means that the children by themselves find a productive form of written presentation of their thoughts. Œuvres, produced in such a way, facilitate easier listening and possible repetitions, if necessary.

"Creative and productive writing" in such a sense is not only a category of native language classes, but in general a platform for reflection on classroom related processes of symbolization. It is not the question if the children should write down something that is correct in the sense of the subject matter, but rather that the children are supposed to find means of a presenting their thoughts which lasts over a longer span of time.

Such classroom culture provides *all* participants with well-founded possibilities to negotiate meaning productively and to produce shared meaning. The specific problem might be to identify forms of externalization which enable all students and (not only) the teacher or researcher to pursue a specific solving process. With regard to arithmetic one can refer here to standardized iconic ways of presentation. However, they need to be assessed and enhanced for this special use of providing reflection for the students.

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CHANGING TEACHING METHODS IN SCHOOL MATHEMATICS

An analysis of some episodes from classes

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Divisibility test were taught to 13 groups of students using four different teaching approaches, two were based on constructivist approach and two on traditional instruction. At the end of the teaching experiment the students' knowledge and understanding was assessed using unseen tasks, two of which were closed tasks, relying on procedural knowledge (i.e. skill acquisition), while two required conceptual understanding. All students performed similarly well on the procedural tasks but students from the two groups whose teaching was based on constructivist principles were more successful on the tasks which demanded the application of knowledge and understanding to unfamiliar contexts.

1. Theoretical framework and related research

In our context it is still a strongly held belief that all the problems with school mathematics can be remedied by a good textbook. Reliable mathematical knowledge can be taught, and students can even learn from a good textbook regardless of their teacher. Underlying this belief is a naïve assumption that the textbook will use the optimal teaching and learning strategy for mathematics (this reveals a belief about the nature for mathematics). If we can discover this optimal strategy and build it into a textbook, it will be possible to teach all students mathematics.

However, current research raises serious doubts about this claim. Counter arguments come from many directions, and the summary of Hiebert and Carpenter (1992) together with more focussed critiques were used to justify our approach.

Constructivist epistemologies (Ernest, 1994, Noddings, 1998, Barab et al., 1999) propose that learning is a dynamic process in which the learners must be active participants. As a result of participating in a community of practice the practices and meanings which develop are far richer, more functional and fundamentally different to teacher or textbook descriptions of those practices and meanings. The role of the teacher is crucial in establishing the appropriate conditions for learner participation, and Simon (1995) and Cobb et. al (1997) have shown that collective posing and solving of mathematical tasks, and teachers facilitation of learners reflections through reflective discourse lead to greater collective knowledge and mathematical development.

Edwards and Keith (1999) discuss the learning mathematics in collaborative small groups. Based on interviews with seven UK students, the benefit of working together as a group, using different skills, listening to and respecting others in the group, helps to build confidence and motivation, and the speed and volume of learning was identified. The result indicate that "students across the attainment range come to appreciate the effectiveness and efficiency of working in such a way" (p. 2-281).

The chosen teaching strategy can influence the form of students' knowledge. In a three-year case study of two schools with alternative mathematical teaching approaches Boaler (1998) found that students who followed a traditional textbook approach developed procedural knowledge that was of limited use to them in unfamiliar situations, while students who learned mathematics through open-ended activities developed conceptual understanding of mathematics that they could use in both school and non-school settings.

It is not sufficient to consider teaching strategies alone as this over emphasises the role of the teacher at the expense of the student. Knowledge and understanding is socially constructed as a result of teacher – student and student – student interaction. Candela (1997) found that students' questions and interventions resulted in the transformation of exercises or demonstrations into problem solving and impacted on the knowledge and meaning constructed from experimental activities. The school curriculum provided a starting point from which the teacher and students reconstructed and elaborated new meanings through negotiation. The study shows elementary school students as active participants in the construction of scientific knowledge.

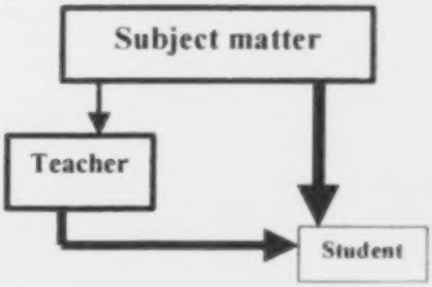
In the study by Yoshida and Shinmachi's (1999) on an experimental teaching program for fractions which was based on an equal-whole schema they found that students who followed the programme understood order and magnitude of fractions better than those who were instructed using the traditional textbook.

When a student works with a textbook, their learning relies on making connections between ideas from the text and their prior knowledge and experience. Britton et al. (1998) propose that making connections depends on the following variables that differ among individuals, for example metacognition, the ability to sense a lack of coherence in understanding and the need for extra connections; inference-making ability; and domain knowledge, which is connected by the inference-making process.

Research into the understanding of cognitive processes involved in learning from text suggests that by enhancing the coherence of texts and the strategies learners use when reading to learn, the quality of learning may be improved (Goldman, 1997).

2. Our research

Fig.1



This paper presents part of an on-going longitudinal study based on the experience of one author who has been directly involved in teaching mathematics at junior secondary level for some years. We have used the following research methods:

a) longitudinal evaluation of teaching effectiveness by comparison of periodic testing of parallel classes,

b) didactical analysis of textbooks,

c) observation of the milieu of the classroom and analysis of teaching strategies,

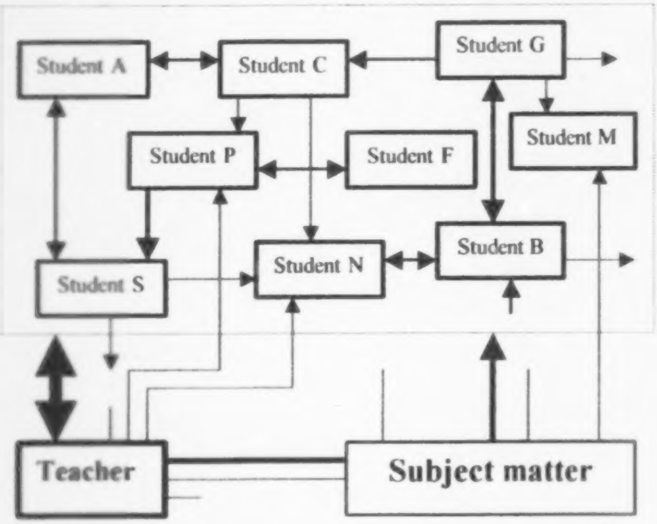
d) the teachers accounts of their own classroom experience,

e) analysis o audio recordings of lessons and children's written work..

These aspects provide a rich source of data from which we can select elements for analysis. It is apparent from our research that the teacher's role is changing. In the recent past, teachers presented information, tested students and imposed systems of tasks prepared beforehand. Nowadays teachers are expected to be organizers and/or activity creators, who use a range of different teaching aids (besides the blackboard), and 'enter the students' space' to co-create a creative climate, context and interpretation of mathematics (Kubínová, 1999). The consequence is that it is necessary to renew the traditional scheme *subject matter – teacher – students* (see fig. 1) and modify the roles within it. It means that (see fig. 2):

- The subject matter is moved to the position of an intermediary which enables the development or modification of already existing students' concepts and the creation of new ones.
- Social relations among individual subjects acting in the process of cognition are accepted which means that the role of the social relationship between students and teacher are accentuated, and the social relations among students are taken into account.

Fig. 2



3. Samples of diagnosis of teaching

The research took place in the school year 1998/99 in thirteen 6th/7th grade classes (12 to 14-year-old students), four of them being situated in a big town. Altogether, 309 students participated in the research. In one class, the teacher was one of the authors of the paper.

Each lesson was recorded using audiotape, students' written solutions of the test tasks were analysed.

The teachers were provided with the following instructions:
Topic: Divisibility tests for 3, 9, and 11
Recommended textbook: (Novotná et al., 1997).
Note: The textbook prefers constructive teaching strategies.

At the end of the experiment, the following tasks were undertaken by the students (teachers had not seen the tasks in advance):
Task 1. Find among the numbers 165, 297, 927, 1 364, 3 183, 8 448, 29 575, 50 578 those divisible a) by 3, b) by 9, c) by 11.
Task 2. Replace □ by the appropriate digits in such a way that the number 28□□6□ is divisible a) by 3, b) by 9, c) by 11, d) by 3 and 11 at the same time, e) by 9 and 11 at the same time.

The results of the experiment were analysed to determine the teaching strategy and the level of understanding of concepts, the use of acquired concept in standard and non-standard problem solving, the ability to transfer acquired knowledge and skills into related areas.

Based on the analysis of the audio tapes the teaching strategy was classified in four ways:

IR: Instructive teaching method, direct teaching of ready information or learning from text.

ID: Instructive teaching method, attempt for students' independent transfer of acquired knowledge.

CR: Constructive teaching method, learning from text.

SC: Social constructive teaching method.

The groups can be illustrated by extracts from typical protocols.

Group IR (7 classes)

Teacher: *You know already the rule for finding whether or not a number is divisible by 3 or 9. An analogous rule is valid for the divisibility by 11. You find it in the textbook on page ...*

Note: In our sample, the cases where the teacher communicated the criterion directly were nearly equally numerous.

Group ID (3 classes)

Teacher: *You know already the criterion for divisibility by 2, 3, 4, 5, 6, 8, 9, 10. Lets try to discover this for divisibility by 11.*

Student A: *It could be similar to 9 and 3.*

Teacher: *Let us try it.*

Student B: *27.*

Student C: *But it is not divisible by 11 while it does not have the sum of digits divisible by 11.*

Student B: *So, 29.*

Student D: *But it is not a number divisible by 11.*

Student C: *So, it does not work.*

Teacher: *It will be best if you look in the textbook, page ...*

Group CR (2 classes)

Teacher: *My compliments, yesterday you did so well when you discovered the rules for divisibility by 3 and 9 using the small cubes. What about a rule for divisibility by 11?*

Student A: *Let us try the same as for 3 and 9. Start for example with 31.*

Student B: *But the sum of its digits is 4 and it is not divisible by 11.*

Student A: *And what about 38, here it is.*

Student C: *It does not work, 38 is not divisible by 11.*

Teacher: *The criterion with the number of digits does not work.*

Student D: *What about tryzing to investigate the sums of digits of numbers divisible by 11.*

...

Teacher: *What have you found?*

Student B: *I reached 165 and the sums are only even numbers.*

Teacher: *What about returning back to our cubes when they were so helpful? Model e.g. 329.*

Student A: *But here, only taking away does not work. For example when I take away one cube from 10, I do not end with a number divisible by 11.*

Student C: *So we add one.*

Teacher: *It is perfect. And what about 100?*

Group SC (1 class taught by one of the authors)

Teacher: *Yesterday when we discovered the criteria for divisibility by 3 and 9, many of you asked about divisibility by 11, but we did not have enough time for it. Has anybody thought of it?*

Student A: *We will calculate the number of digits again.*

Student B: *No, it does not hold already e.g. for 22.*

Student C: *9 was advantageous while we easily took away always one and the rest was divisible by 9. But this already with 10 does not work.*

Student D: *So add one there.*

Student B: *But by 100 we again take one away.*

Teacher: *And what about 1 000?*

Student E: *Hmm, we have to calculate it first ... he nearest is 1 001, we add one again.*

Student A: *But we alternatively take one away and add one.*

Teacher: *Can we express this mathematically?*

Student A: *When we take away a cube from the tens add to units.*

Teacher: *And what about ...*

Student D: *When we add one to the ten to make eleven, we need to take one from the units ...*

4. Analysis of written solutions of the test

Task 1 was constructed to test the acquired procedural knowledge (divisibility tests for 3, 9 and 11), Task 2 the understanding of the divisibility tests and the ability to apply them in non-standard situations.

In accordance with expectations, 89 % of students solved successfully tasks 1(a) and 1(b). Significant differences in the ratio of successful and unsuccessful solutions in the groups did not occur. Task 1(c) was solved successfully by 35,9 % of students (i.e. the divisibility tests were successfully applied). From the table it is seen that students in groups **CR** and **SC** were significantly more successful. Students in groups **IR, ID** often used direct division by 11.

Explanation: The success in the tasks was influenced by two factors:

Applying the divisibility tests for 3 and 9, the student needs to perform only one decisive step (whether the number of digits fulfils/does not fulfil the prescribed condition), while for divisibility by 11 this step is preceded by partial decisions concerning calculation of the alternate sum of digits.

To learn the divisibility tests for 3 and 9, students had sufficient time and practise. The divisibility test for 11 in groups **IR, ID** was set as independent work for students.

Consequently task 2 was non-standard for students in groups **IR**, **ID**.
The results are given in the following table:

Problem	Total	**IR**	**ID**	**CR**	**SC**
1a, 1b	89.0 %	88.4 %	88.4 %	90.9 %	91.7 %
1c	35.9 %	20.3 %	23.2 %	86.4 %	91.7 %
2	22.7 %	8.1 %	8.7 %	72.7 %	75.0 %

5. Conclusions

The view of mathematics which the student has built up during their school career, survives long after they leave school. The situation where mathematics is taught only as a set of precepts and instructions which have to be learnt leads to ever deeper formalism in the teaching of mathematics, resulting in a lack of understanding of the conceptual structure of the subject and an inability to use mathematics meaningfully when solving real problems (Novotná, 1999).

Cultivation of the student's mental representation of world is possible only by deepening his/her active interest in the subject. It is commonly known how difficult this process is given the range of students' attainment, the demand of the subject matter and the fact that student's interest may not be naturally disposed to the subject. Teachers need to be mindful of the importance of the social climate within the classroom and the possible contribution to the quality of students learning.

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Development of concepts for division in third grade teaching experiments: From the viewpoint of the dual nature of concepts and symbolizing processes

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Abstract

We conducted teaching experiments for division in the third grade. In these teaching experiments we intended to develop and analyze students' division concepts, develop sequences of students' activities and develop a framework for analyzing learning and teaching processes in the classroom. In particular, symbolizing processes are emphasized in teaching. Symbolizing processes consisted of process I, from action to drawing pictures and process II, from pictures to mathematical expressions.

In this article we discuss the development of students' division concepts from the viewpoint of the dual nature of concepts and symbolizing processes. Data from pre- and post-interviews are analyzed. Students who have structural concepts for division engage in symbolizing process II with symbolizing process I as a basis. Students who have process concepts for division engage in symbolizing process I and II as independent processes of each other.

Division is introduced in the third grade in our country. In this introduction, we use both quotative and partitive situations. In quotative situation the number of unit is asked, "There are 18 marbles. Distributing three marbles for each child. How many children are there ". In a partitive situation the size of unit is asked, "Twenty-four marbles were shared equally among 3 children. How many marbles did they each have?" One situation is more suitable than another to teach the division concept, but we have not gathered enough student data to discuss teaching methods.

We conducted teaching experiments with several students to investigate the development of students' division concepts, the sequence of activities for students, and a framework for analyzing the learning and teaching process. In this paper, we focus on the results of student interviews and discuss the development of their division concepts.

Background

Kouba (1989) and Mulligan and Mitchelmore (1997) investigated young children's intuitive models of division. Mulligan et al. identified four intuitive models

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for division (direct counting, repeated subtraction, repeated addition, and multiplicative operations) and differentiated intuitive models from semantic structures of word problems or situations. Students use a different set of intuitive models, which they can apply to both multiplication and division problems of various semantic structures. In other words, the same intuitive models can be used for all semantic structures. They insist that there needs to be a clear distinction between the equivalent group semantic structures and the repeated addition intuitive model.

They also discuss the reason why the same intuitive models can be used for all semantic structures. The reason appears to lie in the fact in every multiplication and division situation, " there must be equal-sized groups". Steffe (1991) also mentioned and emphasized the notion of constructing an awareness of unit.

> As a concept, quotative division would have involved an awareness of the result of making, say, units of three using the 21 individual units, a unit containing three, and a method for finding the numerosity of the containing unit before counting. (P.180)

Thus, our teaching experiments focused on the development of students' conceptions for division from the viewpoint of students' awareness of unit and strategies for calculation. We differentiate students' conceptions from the semantic structure of word problems or situations.

Outline of the teaching experiments

The teaching experiments for division consisted of pre-and post-tests, pre-and post-interviews and 8 one-hour lessons. Throughout these lessons we emphasized the concept of unit and the development of calculational strategies. In particular, symbolizing processes were introduced to students in order to support the development of their concept of units and calculational strategies.

We describe two symbolizing processes (process I and II) as they relate to unitizing and calculational methods. In process I, units are symbolized as pictures and in process II, as abstract units (i.e., factors of multiplication or mathematical expression in division). A calculational method is also symbolized in process I as the number of units in pictures and as multiplication in mathematical expression or utterances in process II.

As an example, consider the following division situation that was presented in a word problem. "There are twelve candies. We distribute them three for each child. How many children are there?" Some students explained the quotative situation themselves, other students and a teacher with marbles and the movement of them. Students made four groups of three marbles. The teacher asked students to draw their processes of distribution on paper. A student explained this distribution process by drawing a picture of scattered marbles with his hands. We helped students develop their pictures to solve this problem situation, as seen in Fig. 1. Students were encouraged to interpret this picture in terms of the original problem situation. As a result, students got the answer as four units containing three individual units. With regard to symbolizing process I, students were expected to interpret distribution activities with a picture signifying the result of the distribution. They could get the answer as the number of units in a picture. Drawing pictures was also considered a calculational method for students.

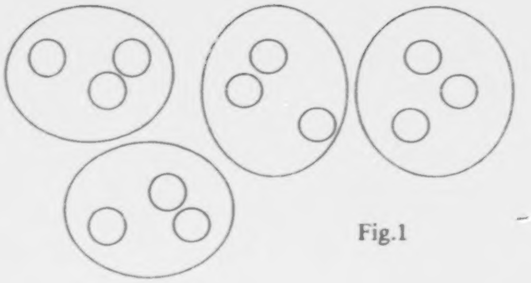
Fig.1

In the second process of symbolizing, we presented the mathematical expression, "12 divided by 3" and a method for calculation as multiplication. Students were asked to interpret the mathematical expression in terms of their pictures and the original problem situation. As students used their pictures and repeated subtraction, multiplication as a calculational method was determined valid. Students focused on units and made a connection between multiplication and division situations.

Students' sequences of activity for symbolizing are organized in 8 lessons as follows in Table 1.

Table.1

	Title of activity	Student's activity , teacher's activity and problems
First	Distributing twelve marbles	Distributing 12 marbles in various ways. 3-3-3-3, 4-4-4, 1-2-5-4 and so on
Second	Formulation of situation for division	Emphasizing equal units with word 'proper distribution' to introduce division situation.
	Definition of division with quotative situation	There are 18 marbles. Distributing three marbles for each child. Distributing 18 marbles and drawing a picture Teacher shows a mathematical expression.
Third	Division in quotative situation	Distributing marbles, drawing a picture, writing a mathematical expression, and calculating with distributed marbles or a picture.
Fourth	Validation of calculational method for multiplication	Validating calculational method with a picture and repeated subtraction. Calculating with multiplication fact.
Fifth	Exercise for calculation	Calculating division problem without situation with multiplication fact
Sixth	Division in partitive situation	Twenty-four marbles were shared equally among 3 children. How many marbles did they each have? Distributing 24 marbles and drawing a picture Teacher shows a mathematical expression.
Seventh	Division in partitive situation	Distributing marbles, drawing a picture, writing a mathematical expression, and calculating with distributed marbles, a picture, or multiplication.
Eighth	Solving division problems	Solving division problem, quotative situation and partitive situation

Data and the outline of interview

Based upon a pre-test with paper and pencil, fifteen students were chosen and

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interviewed. After 8 lessons, thirteen students, who participated in the pre-interviews, were interviewed. Twelve students' data are available.

Interview items are as follows.

Pre-interview

Distributing marbles: quotative situation, partitive situation

Distributing hidden marbles: quotative situation, partitive situation

Post-interview

Distributing marbles: quotative situation, partitive situation

Word problems: quotative situation, partitive situation

Calculation: problems solved with recalled multiplication, problems solved with derived multiplication.

Results of interviews

In this section we will outline the results from pre- and post-interviews.

Pre-interview

Results of pre- interview, which is distributing marbles, are summarized in the Table 2.

Table 2

	Quotative situation	Partitive situation	Subjects
Students' strategies	Quotative strategy	Quotative strategy	Kusano, Niinuma, Arai, Fujii
	Partitive strategy	Partitive strategy	Mori, Iwasaki, Yamada, Moriyama
	Quotative strategy	Partitive strategy	Horiuchi, Kaneda
	Multiplication	Multiplication	Kuniyoshi, Mizunuma

Four students engaged in one quotative strategy in both quotative and partitive situations. Another four students also engaged in one partitive strategy in both situations. Thus, eight students engaged in only one strategy even in different two situations. Two students engaged in two strategies in each situation. Even in the beginning of teaching division, students engaged in one strategy for distributing marbles.

In distributing hidden marbles, we can find instances of symbolizing used by students. I will show a picture drawn by a student Arai.

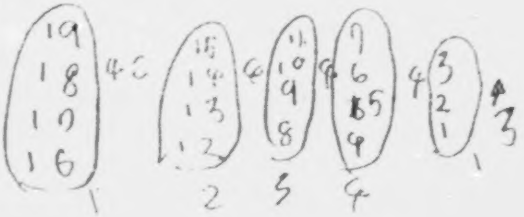

Fig.2

In the quotative situation, "There are 20 candies. Could you distribute four candies for each child?" A student, Arai, drew a picture as in Fig.2. He wrote the numbers 19, 18, 17, 16, then encircled four numbers with line. He repeated this procedure until he wrote the number 1. He wrote the number of individual units, 4, 4, 4, 4, 3 beside each circled portion and the number of units containing four individual units, 1, 2, 3, and 4.

Most students drew almost the same pictures as in Fig.1. They also circled individual units to create new units, and some of them counted the number of units containing four marbles. Students who engaged in a partitive strategy also drew their picture. At first they drew circles in which to put individual units or drew a point to connect lines with individual units. Students were aware of units and equal sizes in a distributing situation.

Post- interview

The results of the post-interview are summarized in Table 3.

Table 3

Calculational strategy Distributing strategy	Recalled multiplication facts and pictures	Recalled multiplication and pictures and derived multiplication facts.	Derived and recalled multiplication facts
Quotative strategy	Arai, Fujii, Kusano, Horiuchi	Niinuma, Moriyama	Mizunuma
Partitive strategy	Kaneda, Iwasaki, Yamada	Mori	Kuniyoshi

Seven students (Arai, Fujii, Kusano, Horiuchi, Kaneda, Iwasaki, and Yamada) used recalled multiplication facts and pictures to solve problems. In problems requiring derived multiplication facts, they engaged in drawing pictures. Their calculational strategies depended on the size of the numbers in problems. Two students (Mizunuma and Kuniyoshi) devised derived multiplication strategies to solve such situations. They used only multiplication in every situation.

In distributing marbles or drawing processes, students used only one strategy. One group of students engaged in a quotative strategy and other students in a partitive strategy. They engaged in one distributing strategy in all semantic problems even in the post-interview.

Discussion

In this section we will closely analyze data from the interviews and discuss the students' concepts of division from both the viewpoint of the dual nature of concepts (Sfard, 1989, 1991) and symbolizing processes (Cobb et al, 1997).

Mulligan et al (1997) point out that intuitive models are available for various semantic structures. In our interview, students interpreted only one semantic structure in the proper way at the beginning of division lessons. In the post-interview, most students also had only one way of interpreting situations. In particular, most students engaged in one way of interpretation in calculational problems in the post-interview. Their way of interpretation had not changed even though they experienced both semantic structures in teaching.

Students changed their calculational strategies depending on the size of the numbers in problems as Mulligan et al (1991) also pointed out. We closely analyzed students' activities in the post-interview. A student, Mizunuma, did not change his calculational strategy in every situation. He devised a new multiplication problem in a problem that was presented with a mathematical expression, 160 ÷ 20 (see Fig.3).

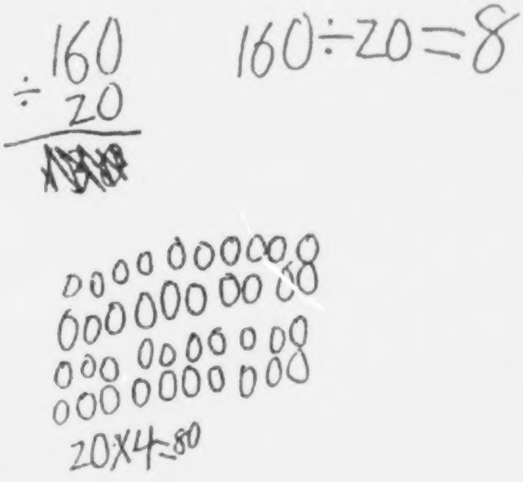

Fig.3

Mizunuma: (He begins to write this in column form.)

160
÷ 20
130

He stopped thinking. An interviewer intervened and asked, "Can you draw a picture?"

Mizunuma: (He gives a slight nod. He begins to draw a picture, draws 20 small circles and draws 20 circles again. There are forty small circles. He suddenly stops drawing circles. He writes a mathematical expression, 20X4=80. He thinks almost twenty seconds. Then, he deletes his answer, 130, in column form.)

Interviewer: How did you think?

Mizunuma: It is a perfectly wrong solution.

Interviewer: It is a perfectly wrong solution?

Mizunuma: One hundred sixty divided by twenty is eight.

Interviewer: How did you think?

Mizunuma: Multiplying twenty by two makes forty. I continued like this and then, multiplying twenty by eight makes one hundred sixty.

Mizunuma got the answer with derived multiplication in a two digit multiplication situation. He drew only forty circles. He did not draw all 160 circles. He used his picture and drawing in such as way that leads us to believe he was aware of units for multiplication. He is aware of units before drawing all circles. The awareness of units helped him curtail his solution procedure for multiplication. For Mizunuma drawing a picture is not only a process but also stands for objects. His concept of division has a dual nature.

Other students drew all 160 circles. One student made mistakes when counting circles and got the answer seven (Fig.4). She used her drawing only as calculation method. Drawing a picture was a process for her.

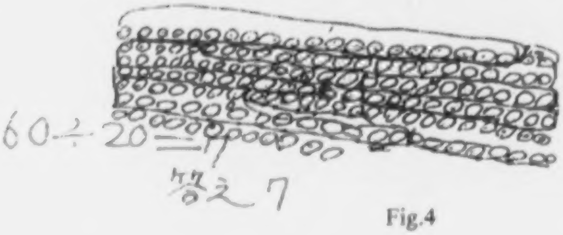

Fig.4

We can see many examples that have process features. Arai also used a picture as calculation method (see Fig. 5.)

He combined individual units to make a bigger unit. At first it was complicated to draw combined units as a unit, but he challenged himself to draw combined units and got the answer eight. He could develop a new unit with the picture but not relate it to multiplication., though an interviewer asked him to relate his picture to multiplication facts. He used pictures as objects, in part. From the viewpoint of the dual nature of concepts, changing strategies for calculations are dependent on students' understanding of the division concept.

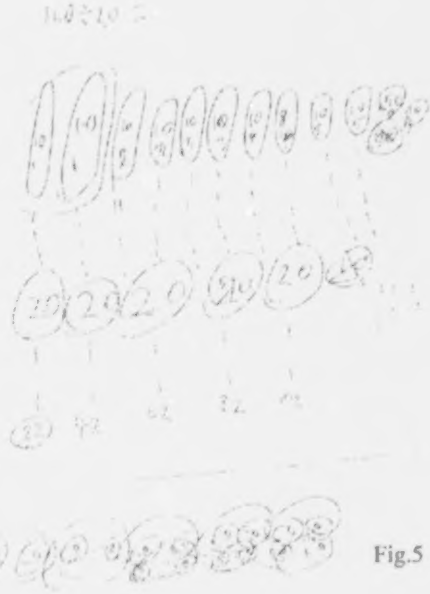

Fig.5

Symbolizing process

The two phases of symbolizing processes are emphasized in our teaching experiments. Symbolizing seems to help some students develop their concepts for division. But there are some difficulties with symbolizing; we will show some of them. Symbolizing processes made explicit during our teaching experiments include two processes: from actions to pictures and from pictures to mathematical expressions.

In symbolizing process I, from actions to pictures, Arai drew marbles. It is difficult for students to separate their actions from the results of their actions. Pictures that we expected students to draw were expressions of the result of actions, not a process of actions. During the teaching episodes, we asked students to answer problems focussing on the results of their actions. This also helps students use pictures as calculational methods.

In symbolizing process II, from pictures to mathematical expressions and multiplication, many students have difficulty relating mathematical expressions and multiplication to their pictures. As we showed, students changed their calculational strategies depending on the size of the numbers in the problems. In problems, requiring derived multiplication, students wrote mathematical expressions in division, but did not calculate with multiplication. They obtained the answer by drawing pictures. In problems requiring only multiplication facts, students seemed to have two calculational methods independently. They did not necessarily relate two calculational methods to each other in these situations. For these students symbolizing process II is an alternative symbolizing process. Symbolizing process I is not a basis for symbolizing process II. For students, problems requiring derived multiplication are first opportunities to relate the two symbolizing processes.

Students who have structural concepts for division engage in symbolizing process II with symbolizing process I as a basis. Students who have process concepts for division engage in symbolizing process I and II as an independent processes of each other.

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Moving between Mixed-Ability and Same-Ability Settings: Impact on Learners

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In this study we analyze why conditions for processes that generate cognitive gains are present in our TAP settings while absent in tracking and study how these processes may affect students' attitudes. During one school year students in four seventh-grades, who learned mathematics both in mixed-ability and tracked settings, were followed. Our observations indicate that there are fundamental differences between interactions in these settings that may affect students' achievements and attitudes.

We endorse many researchers' definition of "Ability-grouping as any school or classroom organization plan that is intended to reduce the heterogeneity of instructional groups; in between-class ability grouping the heterogeneity of each class for a given subject is reduced, and in within-class ability grouping the heterogeneity of groups within the class is reduced" (Slavin, 1996, p.168). Thus a mathematics classroom divided into small homogeneous groups (henceforth 'within-class' tracking') is categorized as tracking. In his comprehensive survey and analysis of tracking studies, Slavin (1996) did not find any significant benefits of tracking. On the contrary, there is a large body of evidence of the harm emanating from tracking. It has been shown that there a gap is created between learners, allocated to different ability-grouping levels, beyond what is to be expected on the basis of the initial differences between them (e.g. Kerckhoff, 1986; Cahan & Linchevski, 1996). On the other hand it has been demonstrated that it is possible to create a mixed-ability (henceforth MA) learning environment in which the growth in inequity, as manifested by the above-reported widened gap between learners, is avoided (The TAP Project, Linchevski & Kutscher, 1996). Moreover lower- achievers' achievements improve while higher-achievers' achievements are not impaired, when compared to their peers learning in the tracking-system (Linchevski & Kutscher, 1998). Thus learning in MA groups may benefit the lower-achievers without being at the expense of the higher-achievers.

Regarding students' attitudes to MA and tracked settings, high-achievers' attitudes seem to undergo change as a result of their learning experiences in the different settings. It has been found that while studying in MA settings all but the highest achievers prefer to study in MA settings; the highest-achievers prefer to change their setting to tracks (Kutscher, 1999). What research has also shown is that when students who have studied in MA settings, continue their studies in tracks: Most students, including 1/3 of the top-track, prefer to revert to learning in MA classes (Boaler, 1997).

In most tracking-studies there have been some attempts to explain the detected effects of the various settings on students' achievements and attitudes. However, there has not been close examination of the processes that lead to these effects. The purpose of the present study is: a) To analyze why the necessary conditions for processes that generate cognitive gains are present in our TAP MA settings while they are absent in the tracking system and to examine processes of interaction between students in the light of this analysis. b) To study how these processes may affect students' attitudes.

Theoretical Background: Interaction studies have shown that active interaction between incompetent and competent students yield cognitive gains both to the less competent students (Botvin & Murray, 1975) and to the more competent students (Doise & Mugny,

1978) when conditions for productive argumentation are met: There is disagreement between the students, the students are strategic (ability to argue/reason the strategy), and the strategies they use are different. Schwarz et al (in press) showed that when these conditions are fulfilled and active hypothesis testing of the task-solution is available, even when the two interacting students are incompetent, at least one of them makes cognitive gains. In sum, all levels of interacting students may gain cognitively provided certain conditions are met.

The required conditions for productive argumentation between incompetent and competent students are intrinsic to the MA learning environment. During the interaction processes between them, competent students and incompetent students frequently disagree. In the resulting argumentation, the competent students are generally strategic; the incompetent students often appear non-strategic because of their difficulties in reflecting on their solution process. The competent students instinctively probe the incompetent students' thinking processes for the reasons of a given statement, thus transforming the incompetent students' explanations to strategic ones and the argumentation to productive. Since students are now able to verbalize their strategies, their argumentation results in their thoughts being organized clearly so that they can analyze their own strategies and pinpoint their differences. The strategy now becomes part of their analytical realm and thus each student gains cognitively.

On the other hand, in school reality it is almost impossible to design a lower-track environment that would meet the above-specified necessary conditions for productive argumentation between mathematically less-competent students. A design of this sort would necessitate first identification of students whose incorrect strategies for solving future tasks would be different. This implies pre-testing the students and analyzing the results before the topic is approached in order to construct small groups with the required profile of students. Moreover, in the case of lower tracks, the condition that both interacting students are strategic is rarely fulfilled. Although these students often do have strategies, they are usually non-strategic: They are unable to fully verbalize their own strategies, nor to dispute each other's arguments and thus their interaction is not argumentative.

Taking the above considerations into account, it seemed to us that in the MA environment conditions for all students to gain cognitively through argumentation exist naturally, while in the tracking system they do not.

Regarding the attitudes of lower achievers, when the processes of interaction yielded constant gains favorable attitudes were expected toward the settings in which they were learning. Thus, among the lower achievers favorable attitudes towards MA settings and unfavorable attitudes towards tracking were predicted. As for the highest achievers, our conjecture was that where these students felt their cognitive gains to be greatest, their attitudes towards those settings would be more favorable.

Study Design: A junior-high school in Jerusalem, Israel, tracked its students in mathematics from the eighth grade and upward. Although the teachers in this school believed in tracks they still felt that it was unfair to track students for mathematics in the seventh grade because these students were new to the school. Thus the school implemented MA classes in the seventh-grade. The school requested a TAP[1] counselor to assist them implement the MA settings according to TAP principles. In this school, at the end of the school year all seventh-graders are tested on a common test and are consequently divided into three tracks according to predetermined cut-off points. As previously stated, this research's purpose was to study processes of interaction in MA and tracked settings and how they influence students' learning and attitudes. Two experiment designs were considered. The first possibility was to study the seventh-graders in their MA settings and to continue studying them when they learned in their eighth-grade tracked class. The second possibility was to study the seventh-graders in their MA settings and then, later on in the same school year, convert the setting from MA to within-class tracking and continue studying the seventh-graders in this tracked setting. Since we considered that the latter experiment-design held more factors constant than the former design, factors such as physical and social environment, teachers and teaching strategies, we thought the latter experiment design more appropriate. On approach, four TAP teachers showed interest in the results of such an experiment. Since the school anyway tracks its mathematics students in the eighth grade they agreed to participate in the experiment and organize their classes for within-class tracking during the last quarter of the year.

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[1] TAP: Together and APart – a project for learning mathematics in mixed-ability classes.

The research population was the students in four seventh-grade mathematics classes. The teachers organized their MA classes for small group-work. They prepared their teaching plan for the whole year at the beginning of the year - topics, textbooks, teaching strategies and so on – taking into consideration the heterogeneity of the student population. During the first phase of the experiment (henceforth 'the first phase') the children learned mathematics in small MA groups and during the last quarter of the school year (henceforth 'the second phase') the children learned mathematics in small homogeneous groups (within-class tracking). Cooperative learning through peer interaction was encouraged in all learning environments. The same quality of learning materials, learning strategies and class culture was fostered in both phases of the experiment. Before the second phase, the teachers assigned their students, without disclosing this to the students, to three mathematics tracks as they conceived the students would be assigned the following year. An 'H' student would most likely learn in the highest track in the eighth grade, an 'M' student in the middle track and an 'L' student in the lowest track. The researchers observed the classes for 60 hours. Seventeen of these hours focussed entirely on collecting data of within-group interactions – nine hours while the students learned in MA settings and eight hours while they learned in within-class tracking. A sample of students from each of the three tracks, 35 in all, were interviewed twice: once just before the class was reorganized into within-class tracking and once at the end of the school year. Each interview was semi–structured with 12 prompts. The interviews were recorded on audio tapes and transcribed in full.

Results and Discussion

Students' Learning in MA Settings: The cooperative-learning culture fostered in the research classrooms was one where students generally solved the tasks independently and thereafter were expected to present their solutions to their group-mates and resolve any disagreements through argumentation. However when a student experienced difficulties with a task s/he was engaged in, s/he was encouraged to discuss and try to resolve this difficulty with a group-mate. Thus a culture of mutual responsibility and support was fostered in these groups.

The following is a typical example of this sort of peer interaction in a MA group. The students were working on the concept of 'solution of an equation'. The current task called for the construction of equations according to given constraints. For example: "Construct an equation with *m* to represent its unknown, –2 is *m*'s coefficient, and 3 is the equation's solution." The teacher had taken great care to explain the notion of solution of an equation prior to this task. Six students made up this MA group: S_H (H-student), L_M (M-student) and Z_L (L-student) were three of the six students working in the group. (The student track-levels were added only during the write-up of this paper on the basis of the results of the end-of-the-year tracking-tests.) S_H was working with students on her right, while L_M, who was sitting on S_H 's left, was working with Z_L. They had just solved the first item (albeit incorrectly), and were attempting the second.

L_M (reading the task with Z_L, sitting on her right): *Now "An equation with unknown 'a' and the operation of addition, so that it's solution will be 0'. So let's do -5+a=0. And: Check -5+5=0.*

And they proceeded to the third question completely unaware of their misconception. S_H, turned to see what L_M was doing and noted that for the first question that required "An equation with unknown *x* whose solution is –4", L_M had written *4x=-4*

S_H (says assertively): *No, it isn't right.*

L_M: *Why?*

S_H: *'x' is equal to -4, the 'x' is equal to -4*

L_M: *But it doesn't matter, anyway I did it -4 times, like, 1*

S_H: *But no, but the x is equal to -4*

L_M: *Oh, I understand* (L_M erased her old answer and wanted to write a new one) *Now let's see, what will we do? First of all x* (and writes '*x*')

S_H: *The solution doesn't have to be at the equal, in the beginning. So what do you want to do? x plus or minus or multiplication?*

L_M: *Multiplication, but I'll write it after it.*

S_H: *So x times*

L_M: *2* (L_M now has written '*x•2*')

S_H: *Okay, times 2 equals?*

L_M: *-4*

S_H: *Minus what?* (S_H holds L_M 's hand to stop her from writing) *'x' equals to -4! -4 times 2?*

L_M: *Just a minute. So I've done everything wrong. Oh, I thought that the final solution has to be -4.*

S_H: *No, "whose solution", that the unknown is -4.*

L_M: *So that's it, I thought that the solution at the end* (meaning that the number should comprise the right-hand side of the equation) *has to be -4.* (L_M verbalizes her strategy)

S_H: *-4 times 2 equals?*

L_M: *-8* (And L_M completes her equation to '*x•2=-8*')

S_H: *Check!*

L_M: *-4 times 2 is equal to -8* (Writes this down). *Now I have understood. Just a minute, then I'll erase everything. Oh no!! Oh no!!*

L_M proceeded to erase all her solutions that she now realized were incorrect, and to re-solve the tasks correctly on her own while S_H watched over her until convinced that L_M really understood. L_M now turned to Z_L and engaged in a similar argumentation with her, essentially repeating the same process that she had just undergone with S_H. Through L_M's discourse with S_H, L_M realized that she had misunderstood the notion of 'solution'. Even though the students had been exposed to the new definition of solution, for a while it seems that for L_M, the notion of solution as an 'answer' to a numerical string was dominant. Through S_H's continual argumentation, L_M became conscious of her strategy and was able to verbalize it - L_M had become strategic. L_M was able to analyze the source of her misconception, as is attested by her discovery: *"So that's it, I thought that the solution at*

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the end has to be -4", and thus gain cognitively. L_M was now able to set aside the 'old' notion of solution and to strengthen her 'new' notion of solution. Not only did L_M have the opportunity to clarify her misconception but, in the subsequent interaction with Z_L, L_M enabled Z_L to be strategic and in this process further refined her own understanding of the concept.

Students' Learning in Tracked Settings: All students sat in homogeneous small groups continuing the same cooperative learning-culture as had been nurtured in the MA settings. Although at no time did the teachers hint that the new groups were tracked it was evident to the students that it was indeed so. The teachers reported that the H-tracks functioned wonderfully, the M-students were trying hard but very often got stuck on a task and were unable to resolve their difficulties with the human resources in the group. As for the L-track, they barely managed anything on their own, and the teachers found themselves torn: Generally, if they didn't monitor the L-students, the latter accomplished very little; and yet M-students needed them too. The following is a typical example of M-students struggling within their group to solve a given task. This class was revisiting the notion of substitution[2] in algebraic expressions and all tracks were engaged in the following task:

$3a$ is the given algebraic expression. Write next to each numerical expression, given that it was derived from this expression, what number/expression was substituted?

1) $3\cdot 8$ __________ 2) $3\cdot(-5)$\__________ 3) $3b$ __________ 4) $3(7+5)$ __________

Five M-track students, T_M[3], A_M, Y_M, S_M and J_M are trying to work on the first item, but no one really understands. R – the researcher:

A_M: *Ah, I know. The fact that they say, let's say, these sentences* (maybe referring to the task instruction "what number/expression was substituted") *that: 8 is the sum. 3 times 8 something like this.*

T_M: *It's 3.*

A_M: *You're wrong* T_M, *it's not 3. You're wrong you have to write a sentence.*

Y_M: *You need letters here. That's what I know, that you need letters.*

J_M: *I think that you have to substitute instead of the "a" the number 3.*

A_M: *One can do 'a times d' and* $a=3$ *and* $d=8$.

J_M: *It could be right.*

Discussion ends. The researcher sees that the discourse has come to a dead-end and calls G_H from the H-track to see what would transpire:

R. G_H, *please come here.*

G_H joins the group. Productive argumentation is initiated, similar to the one described in the previous section. When the researcher sees that the group is able to answer the first two items correctly, she sends G_H back to her H-group, and observes the M-group continue working on the rest of this task:

R: *Right.* S_M, (who is the weakest in the group and did not even try previously to participate or make sense of the task) *what will be the answer in question number...* S_M, *what will be* (the answer to) *item 3?*

S_M: *a equals b.*

A_M: *What?*

J_M: *To b.*

S_M: *To b.*

A_M. *A letter is equal to a letter? What's these nonsenses?*

J_M : *A letter is like a number, but* (it's an) *unknown.*

R: A_M, *do you accept this?*

A_M: *No, I don't think so.*

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[2] One of the principles of TAP is to revisit concepts spirally.

[3] Naturally the students referred to their group-mates by name and not by level. We attached the levels to their "names", according to the results of the end-of-year tracking test, to facilitate reading.

R: (turning to T_M): *Can you explain to him?*

A_M: (Frustrated with not being able to understand and unwilling to address this issue further): *Alright, I understood.*

T_M: *Four* (meaning item four): *a is equal to 12.*

R: A_M, did you understand the idea about 'a equals b'?

A_M: *No, it's alright.* (Item) *four is either 'a equals 12' or 'a equals 7+5'.*

R: *Right, nice.*

All students in this M-group were serious and motivated students who, when learning previously in their MA small groups and when they felt they needed had asked for help from their group-mates, and had consequently been able to tackle most tasks. Before student G_H joined the group the students were making no headway. There was disagreement but the students were not strategic: They could not offer reasons or arguments to defend their solutions nor to dispute other students' solutions, thus no learning evolved. With G_H's argumentation the picture changed. G_H was able to articulate her arguments very lucidly, was certainly strategic and enabled the M-students to analyze and clarify their thoughts. All members of this M-group seemed to gain a clear understanding of the concept involved in the task. Student A_M was a very diligent student and during the first phase had been able to resolve most cognitive conflicts within the framework of the heterogeneous group. Clearly this exchange shows that he was getting frustrated: He was unable to grasp that one could substitute a variable for a variable. In contrast with the cognitive gains that occurred through the interaction between G_H and the group, for A_M the interaction dealing with this concept proved unproductive, unfruitful, frustrating and apparently yielded for him no cognitive gain. This might have occurred since the present constellation of students were unable to help A_M become strategic and enter into a productive argumentation with him.

It seems that, even after a cooperative culture of argumentation has been established, it is very difficult for lower-track students to engage in spontaneous productive argumentation without the input of external sources. These sources might be specially designed learning materials, a teacher, a more competent student and the like. What has been demonstrated is that in a MA environment a cooperative culture of productive argumentation can be established quite naturally.

Attitudes to learning in MA and tracked settings: The students were interviewed at the end of the first and second phase of the experiment. The results will be divided into two categories: Those students who would eventually be assigned to lower tracks (L and M students: L&Ms) and those who would eventually be assigned to the highest tracks (Hs). L&M's attitudes: In the first phase all the L&Ms consistently preferred learning in MA classes. They credited their progress to the group, emphasizing the mutual support. When prompted, the Ms all valued the opportunity of being able to help the students in their group as a medium for their own cognitive gains. All L&Ms objected to tracking, firstly because of the feelings of shame and failure related to being assigned to the lower tracks; secondly because of the lack of the stronger students, thus depriving them of progress. Some representative quotes:

G_L: (The groups helped him) *Because if I get stuck on something then they help me ... and sometimes I helped children and sometimes they helped me ... Sometimes* B_H[4] *and sometimes* A_H *helped me. Both of*

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[4] We remind the reader that the students referred to their group-mates by name and not by level. We attached the levels to their "names", according to the results of the end-of-year tracking test, to facilitate understanding.

them ... And I helped N_L (and this helped him) *because I helped him and I began also – as if I'm explaining to myself.*

P_M: (The groups help since) *It helps more, that if you don't understand something, and the friend that is sitting next to you also doesn't understand, you also have many more friends that you can ask.* (I usually ask) *Tal_H and Ad_H ... I* (also) *help Mo_L* – (and this helps P_M since) *I go over the material that I'm saying to her, that I'm explaining to her. I know it well every time I explain to her ...* (She prefers learning in MA settings) *Because there, let's say there's the M-track, let's say I will be in the M-track, I think I'm an imbecile. But in the class when you learn together in the same group, and no one thinks that he is less good or better. 'Good' is the same measure* (for all). *And in tracks there is best, less good, and bad.* (And if she'll be in a lower track) *I'll say to myself, why did I try so hard and I didn't do well ... And I won't feel good with myself".*

V_M: *"I don't like this whole business with tracks. If one doesn't understand something, if you put somebody next to him that knows that after 9 comes 10, then he goes and asks him. So he answers him and the boy knows, he learned. But if all the students in the class didn't know what number comes after 9, then simply they'll have a long time to understand, till the teacher will teach them all..* (She prefers) *The class will be with children of a higher level,* (and) *children of a lower level, so that children of a higher level can explain to children of a lower level. When you sit* N_H (a borderline H-student who was in V_M's group in both phases) *next to* Al_H *then she will progress and become like him".*

In the second interview, the L&Ms continued to unequivocally prefer to learn mathematics in MA settings. They now also expressed frustration for two main reasons: Firstly, because they were often (the Ls – almost always) unable to resolve their problems in their groups. Secondly, concerning the tracking that would occur in the eighth grade they were concerned that they might indeed be assigned to the lower tracks in the following year, that in the M-track they would be slowed down, and that they would feel inferior. However, if indeed tracking was unavoidable, their fervent desire was to learn in the H-track and they felt they were certainly capable of it.

P_M: *In the former group* (MA) *there were children that were cleverer ... they were able to help more ... and* (now) *I understand the material less ... In the M-track* (in the eighth grade) *there are many children, let's say, that don't understand the material, so one progresses slower... I want to be in the H-track ... If I will be in the M- track I will think that I'm not clever and things like that.*

H-students' attitudes: Of the 35 students interviewed, after the end-of-year "tracking" test, 18 were eventually assigned to the H-track in the eighth grade. In the first interview eight wanted to try tracks, eight preferred to continue learning in MA settings and two were ambivalent. In the second interview ten preferred tracks (8+2), one (2-1) remained ambivalent, and seven (8-1) preferred MA settings. Generally, these findings support those reported in the literature.

How did the processes of interaction influence their attitudes? During the first phase, all the Hs who wished to continue learning in MA classes spoke of how good it was learning in their groups: the cooperation, explaining to one another, learning through arguing - for instance- "*why they were mistaken rather than what the mistake was*". Those Hs who preferred to try tracking spoke of how they – "*want to progress faster*", even though all were very happy learning in the MA groups, enjoyed the group-work and especially appreciated their opportunity of being able to help others as a means of improving their knowledge. Although they wanted to try tracks some were also concerned about tracking because of the pressure involved, such as learning high-level material at a rapid pace. They felt more secure learning with their homeroom classmates, and felt bad about the stigma of the lower-tracks. During the second phase, most of the attitudes seemed to have been reinforced by their experience in their tracks. However, those for MA, now spoke about the unhealthy competition between the H-students, about the fear of dropping from a higher to a lower track, of too rapid a pace of learning. Those who favored tracks emphasized the fact that in tracks one can progress faster.

It is interesting to note that, unlike reported literature findings, the experience of changing the learning settings from MA to tracks had little influence on students' attitudes: It seems that the change only reinforced their former attitudes. Both in the first and second phase, all L&Ms' preferred learning in MA settings. However, the Hs' attitudes were not uniform: both in the first and second phase, two stable groups could be identified, about half favoring tracks and the other half favoring MA settings. It was only when we saw and analyzed the tracking-tests' results, that we discovered that there was a strong correlation between the results of the H-tracks and their attitudes: The highest-achieving Hs preferred tracks, the lower-achieving Hs preferred MA settings. And indeed cognitive gains seemed to feature strongly in their reasons.

Only two students changed their attitudes with the change in settings, both highest-achievers. In the first phase, Al_H was extremely satisfied with the way they learned: *Liat* (teacher*) gives us freedom* (to progress as they wished)... *It was fun, everybody discussing and working together... All* (his group) *are really on the same level. Only at certain points, some have difficulties...* (When working with them) *I revise the material and I see that I understand it.* In the second phase Al_H now favors tracks: *I think I progress faster in the new group... I learn more, I can ask more, I have more time for myself...* (Although, he saw the benefits of the groups in the first phase) *It helped me. When I explain to others I see that I really understand... In this* (the second phase) *I don't explain so well... they already all understand... It's a pity... If I have* (next year) *someone in my group that I'll be able to help, it will be very good. He doesn't have to be weaker or stronger than me. If I have the opportunity* (to explain) *then I want to do it.*

It would be interesting to see what effect each track would have on these students' attitudes when learning in between-class tracking. In particular, whether the lower and highest Hs would still maintain their previous attitudes.

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A DEVELOPMENTAL ASSESSMENT OF PUPILS USE OF DOMAIN-SPECIFIC AND GENERAL STRATEGIES IN PROBLEM SOLVING

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ABSTRACT

This paper presents findings from a study on assessment of pupils' use of domain specific and domain general strategies in problem solving. Five novel problems were administered to 720 Cypriot pupils and since the problems were novel, pupils had to use their existing general strategies in order to solve the problems. A pilot study revealed three main domain specific strategies that pupils had used and which were reflecting their knowledge in combinatorics. Three general strategies were also identified. The construct validity of the test was examined by investigating the developmental properties of the test and a significant improvement was evident in each performance variable of older pupils. Moreover, the convergent validity of the test was examined via correlations with a national test. Finally, an association between pupils' domain specific knowledge and general problem solving strategies was identified. Implications of findings for the development of a new model for assessment in problem solving are drawn.

I) INTRODUCTION

Recently published literature on curriculum and assessment development in Mathematics has heavily emphasised problem solving (English, 1999). Problem solving is cognitive processing directed at achieving a goal when no solution method is obvious to the problem solver (Schoenfeld, 1985). This definition implies that the debate about the importance of domain-specific knowledge and domain-general strategies in the development of problem solving competence should be taken into account in any attempt to establish a model for assessment of problem solving. Supporters of the domain specificity of problem solving skills frequently underestimate the role of general strategies. Sweller (1990) argues that pupils' skills in problem solving are dependent on schema acquisition and rule automation and general problem solving strategies are of little use. He argues that without automation, novel problem solving is likely to be a difficult or impossible task for the vast majority of people. On the other hand, researchers who emphasise the importance of general strategies argue that the self-regulatory mechanisms contribute not only to immediate performance but also to continue growth of the cognitive system (Scardamalia & Bereiter, 1985).

There are many studies investigating either domain specific knowledge or domain general strategies but few that have examined the interaction of the two. The study reported in this paper was designed to help us develop a model for assessment of problem solving based on the assumption that the components of problem solving are the domain specific knowledge (i.e. content understanding) and the problem solving strategies which are divided into the domain specific strategies and the general strategies (O'Neil, 1999). As far as the general strategies emphasis is given to those strategies that perform a self-regulatory or metacognitive function. This paper illustrates findings of a study concerned with an attempt to obtain construct related evidence of a developmental assessment of pupils use of domain specific and general strategies in novel problem solving. It was also possible to examine whether there is an association between pupils domain specific strategies and general strategies.

II) METHODS

The test used in this study was consisted of problems which require pupils to use both types of problem solving strategies. It was assumed that if children had to solve novel problems and had only informal knowledge of the problem domain, they would be more likely to use general strategies to solve the novel problems. Given that the problem domain is a critical factor in my attempt to examine the relationship between domain specific and general strategies, the development of a suitable set of problems was important. The mathematical topic of combinatorics was chosen as the problem domain for two reasons. Since there are many ways of forming combinations of two sets, the domain is suitable for designing problems that allow for different levels of solution. Moreover, from the developmental perspective, the establishment of a combinatorial system plays a central role in Piaget's theory of cognitive growth. It can be claimed that the problems of the test evaluated drew upon a body of domain-specific knowledge which allow for different levels of solutions. And since problems were presented in a familiar context, pupils could at least attempt a solution by applying their informal knowledge of the domain along with their existing skills in using general strategies of problem solving.

Validity is the key criterion for judging the adequacy of outcome measures. The examination of the construct validity of this test is particularly important since it is closely related to the aims of the study. Various types of information can be collected in order to obtain a complete understanding of the construct underlying a test. Typically, the development and validation of educational assessments involves three critical phases of study: a) conceptualisation of an attribute based on substantive theory, b) definition of an attribute in observable terms such as test items, and c) collection of data to verify that the measured attribute behaves in concordance with the underlying theory (Cronbach, 1990). Thus, an important step in this investigation was to define the parameters of this study. A pilot study was conducted which helped me to identify the conceptual framework of this study. An analysis of pupils' responses revealed that three domain specific strategies were used. The first strategy did not involve any attempt to plan the combinations that pupils could do in order to find all the possible ones. They used to select items randomly and reject those which did not meet the requirements of the problem. The second category was more efficient than the first one but still pupils did not follow an algorithmic approach (i.e. the third strategy) to find the solution. The essential characteristic of the second strategy had to do with the fact that pupils follow a pattern in their attempt to select all possible combinations. It can be claimed that these strategies reflect pupils' knowledge in combinatorics since these strategies involved means of selecting and combining pairs of items. It was also found that most of the pupils followed specific strategies to monitor and regulate their solutions. The strategies could be either one or two dimensional. The two dimensional strategy was concerned with pupils attempt to check their solutions by looking at both variables of the problem. The latter was seen as representing pupils' skills in applying general problem solving strategies. Semi-structured interviews with 8 pupils helped to clarify further the findings of the pilot study. Once the final version of the test was developed, the specifications and the problems were content validated by 7 teachers. Minor amendments were made and the written test was administered to 720 Cypriot primary pupils. The median age in the overall sample was 9.5 years with pupils ranging in age from 5.9 to 12.2 years at time of testing. The total sample consisted of 368 (51%) girls and 352 (49%) boys. The test was designed as an untimed test and classroom teachers administered the test in two forty minutes sessions. Each teacher was given standardised instructions for administering the test.

Pupils' responses to each problem were given a score of 1 to 3. Score 1 was given to those pupils who ignored the question of the problem and gave irrelevant combinations as the solution of the problem, score 2 to those who only found some correct combinations and score 3 was awarded to pupils who answered correctly without any error. Moreover, data were kept about the type of solution strategy that each pupil had used and the type of strategy she/he had used to monitor and regulate the solution of each problem. To examine the reliability of performance variables, the responses of 25 pupils were subjected to two independent ratings and a 92% level of interratter agreement was obtained. Moreover, the reliability of the findings was measured by calculating the relevant values of Cronbach Alpha for the three scales used to measure pupils' performance. The values of Cronbach Alpha were higher than 0.80.

This paper presents findings concerned with three questions of this study. First, the construct validity of the test is examined by investigating the developmental properties of the test. Second, the convergent validity of the test is examined via correlations with national tests designed to measure performance standards in primary mathematics (Kyriakides & Gagatsis, 2000). Finally, the association between pupils' domain specific knowledge and general problem solving strategies is investigated.

III) FINDINGS

A) Validity: Developmental properties of the test

The developmental properties of the test were examined by finding the descriptive analyses of scores from individual problems in terms of the three performance variables using subgroups of pupils sorted by age. Table 1 shows the percentages of pupils of each year group according to the kind of domain specific strategy they used in order to solve each problem. We can observe that the majority of year 1 and year 2 pupils used non planning strategies to solve each problem. On the other hand, the majority of older pupils (years 5 and 6) used algorithmic strategies to solve each of the five problems. Finally, year 3 and year 4 pupils used either non-planning strategies or they follow a pattern in order to find all the relevant combinations of each problem. It is also important to indicate that the percentages of pupils who used non-planning strategies to solve each problem decrease by age whereas the percentages related with algorithmic strategies increase. It was assumed that the scale used to classify pupils' use of domain specific strategies produced ordinal data. Thus, Kruskal-Wallis one-way analysis of variance revealed that the use of domain specific strategies varied significantly ($p<.05$) with age for each of the five problems.

Table 1: Percentages of pupils of each year group according to the kind of domain specific strategy they used to solve each problem

Groups of Pupils	Non-Planning Str. Problems					Following patterns Problems					Algorithmic Str. Problems				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
Year 1 (n=116)	81	78	82	87	89	19	22	17	13	11	0	0	1	0	0
Year 2 (n=119)	78	71	80	81	83	18	24	16	15	15	4	5	4	4	2
Year 3 (n=124)	50	58	60	72	71	44	37	35	25	29	6	5	5	3	0
Year 4 (n=121)	41	45	49	61	59	52	49	44	33	36	7	6	7	6	5
Year 5 (n=118)	11	18	12	23	29	54	46	54	43	41	35	36	34	34	30
Year 6 (n=122)	9	12	10	15	18	35	29	24	23	25	56	59	66	62	57

Table 2 illustrates the percentages of pupils according to which of the three general strategies they used in order to examine the appropriateness of their solutions to each problem. The following observations arise from this table. First, pupils of year 1 and year 2 did not use any strategy to check the solutions they gave. Second, the percentages of older pupils who did not use any strategy to check the appropriateness of their solutions are smaller than those of younger pupils whereas the percentage of older pupils who used the two dimensional approach to check their solutions were bigger than those of older pupils. This reveals a significant improvement in the general strategies that older pupils use. This assumption was examined by using the Kruskal-Wallis one-way analysis of variance which revealed that the use of general strategies varied significantly ($p<.05$) with age for each of the five problems. Third, the percentage of each year group who used each type of general strategy do not vary from problem to problem. This might be seen as an indication that the adoption of a particular general strategy does not depend on the difficulty of the problem. It is finally important to note that the percentages of year 5 and year 6 pupils who did not use any strategy to check their solutions are relatively high.

Table 2: Percentage of pupils of each year group according to the kind of general strategy they used to solve each problem

Groups of Pupils	No examination Problems					One dimensional Problems					Two dimensional Problems				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
Year 1 (n=116)	70	68	70	69	71	21	22	23	21	22	9	10	7	10	7
Year 2 (n=119)	62	61	64	61	63	28	29	25	29	28	10	10	11	10	9
Year 3 (n=124)	41	42	40	42	41	43	43	46	45	49	16	15	14	13	10
Year 4 (n=121)	32	35	33	31	34	49	47	50	52	50	19	18	17	17	16
Year 5 (n=118)	28	29	32	30	29	47	45	44	46	46	25	26	24	24	25
Year 6 (n=122)	25	22	20	19	18	43	45	46	48	45	32	33	34	33	37

Finally, table 3 presents the mean scores and the relevant standard deviations derived from the responses of each year group to each problem. To examine the relationship between age and test score, an ANOVA was performed with age as the independent variable and the mean scores across the five problems as the dependent variable. This produced a significant relationship ($F=12.93$, $p<.001$). Post hoc analysis revealed that pupils of year 6, year 5 and year 4 performed significantly better ($p<.05$) than pupils of year 1, year 2 and year 3. These findings are reflected in Table 3 where an increase in older pupils performance in each problem can be identified. It is also important to note that the majority of younger pupils could not give even a partly correct answer.

Table 3: Mean scores and standard deviations for each problem by year group.

Groups of Pupils	Probl. 1		Probl. 2		Probl. 3		Probl. 4		Probl. 5	
	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
Year 1 (n=116)	1.42*	0.89	1.32	0.84	1.33	0.76	1.24	0.71	1.19	0.68
Year 2 (n=119)	1.41	0.91	1.62	0.88	1.35	0.82	1.28	0.87	1.22	0.72
Year 3 (n=124)	1.64	0.89	1.72	0.90	1.53	0.96	1.54	0.81	1.59	0.78
Year 4 (n=121)	1.94	0.92	2.02	0.84	1.83	0.99	1.84	0.81	1.89	0.98
Year 5 (n=118)	2.14	0.87	2.28	0.82	2.03	0.92	2.04	0.81	2.02	0.68
Year 6 (n=122)	2.28	0.80	2.41	0.66	2.13	0.91	2.24	0.81	2.18	0.68

* 1= Irrelevant combinations or no solution, 2 = partly correct answer, 3 = correct

B) Convergent validity

Table 4 illustrates correlation coefficients using scores of each performance variable and convergent validity coefficients with the total and domain scores of a test used to measure performance standards of Cypriot pupils in primary mathematics. The figures of Table 4 revealed strong relationships between the test score in combinatorics and the score of each domain of the national tests but that of Geometry and Measurement. Moreover, the relationship between each domain with the score derived from the domain strategy that pupils used to solve the problems are also very strong but still the relevant correlation of the domain of Geometry and Measurement is relatively small. This implies that pupils who managed to achieve higher scores in the national test on each domain of mathematics which is related to combinatorics are those who used more sophisticated domain strategies. We can also observe that the correlation of the general strategy with the domain of Geometry is as big as those of the other domains with the general strategy. This implies that there are significant relationships between the general strategies and the score of each domain of the national test. Finally, it is interesting to note that among the correlations of the four domains with each of the performance variable of the test the biggest are those of the problem solving domain.

Table 4: Convergent validity coefficients

	Domain Strategy	General Strategy	Test Score
Combinatorics test			
Domain specific strategy	1.00		
General strategy	0.44	1.00	
Test score	0.65	0.51	1.00
Performance Standards			
Arithmetic	0.48	0.41	0.52
Data Handling	0.47	0.40	0.54
Problem Solving	0.49	0.43	0.56
Geometry & Measurement	0.29	0.38	0.35
Total score	0.50	0.44	0.56

C) Associations between pupils domain specific strategy and general problem solving strategies

The correlation coefficients among the three performance variables of the combinatorics test (Table 4) revealed that there is a statistically significant relationship between the average test score and the score of the domain specific strategy. Thus, pupils who used more complicated domain strategies managed to find more appropriate solutions to the problems. We can also observe that there are statistically significant correlations between the use of general strategies with both the test score and the domain strategy score. However, the values of the coefficients are smaller than those of the domain specific score with the total test score. In order to clarify further the association between pupils' use of domain specific strategy and the use of general problem solving strategy the relevant correlations for each problem by age group were calculated. All these correlations were statistically significant ($p<.05$). It was also found that there was not much disparity between the values of these correlation coefficients since the minimum value was .38 and the maximum was .46. Nevertheless, the values of correlation coefficients derived from analysing older pupils' responses (year 5 and 6) were relatively smaller than those of younger pupils (year 1, 2 and 3).

To clarify further this finding, the relevant cross-tabulations were examined. It was found that a significant percentage of older pupils (13% of year 6 and 10% of year 5) managed to solve the problems correctly without using a two dimensional approach to check their solutions.

IV) Discussion

The evidence presented above can be discussed in terms of its implications for the development of a new model for assessment of problem solving. Suggestions for further research can be also drawn. First, assessment of problem solving should cover not only pupils' abilities to use domain specific strategies but also their abilities to use general strategies. The study seems to reveal that it is not only feasible to assess these two types of strategies but that it is also possible to establish a developmental assessment model and to use relevant techniques to measure the construct validity of the measurement instruments. The fact that both the construct and the convergent validity of the test used for the assessment of domain specific and general problem solving strategies are satisfactory provides the basis upon which a new model for assessment of problem solving can be developed. Moreover, the need of conducting further validity studies on pupils' skills to use these two strategies in other domains of mathematics can be raised. Second, my approach to measure the validity of the test by examining the developmental properties of the test can be enriched by conducting a longitudinal study (Goldin & Passantino, 1996). Such research may help us to compare the developmental properties of the test in different periods and to identify the effect of particular interventions on pupils' abilities to use each type of problem solving strategy. At this period the test is administered again to the same sample of pupils and value added analysis is conducted which could help us to measure pupils educational progress in each of these two types of strategies (Kyriakides, 1999). Third, the content understanding, which is another aspect of problem solving, was not measured in this study. However, the theoretical framework of this new approach for assessment of problem solving reveal that further research is needed to examine the relationship between content understanding and pupils' abilities to use problem solving strategies. The content understanding can be assessed by asking pupils to develop concept maps and this technique is used in a study that is nowadays conducted to examine the validity of this approach for assessment of problem solving.

Finally, pupils' responses to this test have significant implications for the primary mathematics curriculum. The findings of this study reveal that primary school pupils are able to use both domain specific and general strategies in problem solving. The correlation coefficients among the three performance variables reveal that there is a significant association between these two types of strategies and that both of them are fundamental to success in problem solving. This implies that teachers should provide opportunities to primary pupils to use both strategies in meaningful problem solving situations. However, a significant percentage of older pupils did not use general strategies. This may be attributed to the fact that Cypriot teachers do not give particular emphasis in fostering the development of their pupils metacognitive skills (Philippou & Christou, 1999) and this might be one of the reasons for which standards in problem solving are particularly low (Kyriakides & Gagatsis, 2000). Both teachers and pupils need to realise the importance of engaging in self monitoring in order to develop their skills in using not only domain specific but also general problem solving strategies.

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CODING THE NATURE OF THINKING DISPLAYED IN RESPONSES ON NETS OF SOLIDS

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In response to Mariotti's study on the complexity in mental imaging using nets of solids, the authors formulated three research questions in association with the development of a test in 3-dimensional geometry. This paper reports on the coding of responses to a question on nets, using two frameworks., the SOLO Taxonomy and the van Hiele levels of understanding. The findings show that students' descriptions of nets can be coded in both frameworks, thus supporting the three research questions.

The understanding of the nature of growth in students' insight into 3-dimensional geometry continues to interest researchers (Despina, Leikin & Silver 1999; Dreyfus 1992; Gutiérrez 1996; Mariotti 1989; Mitchelmore 1980). As part of this growth in insight, visual imaging of 3-dimensional figures is important in that it promotes the development of understanding of the relationships within a figure. Gutiérrez (1996, p.1-9) defines visualisation in mathematics as the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, and that it contains four main elements, mental images, external representations, processes, and abilities of visualisation. He considers that visualisation is one of the main bases of cognition (p.1-3), while Mitchelmore (1980, p.83) argues that "it is of great value to be able to visualize and represent three-dimensional configurations and to comprehend the geometrical relations among the various parts of a figure." Students' ability to decompose and recombine images, that is the ability to break down a visual image into simpler parts and them recombine those parts into new images, is an important component of imagery (Brown & Wheatley 1997).

Two ways in which the ability to translate between 3-dimensional solids and their 2-dimensional representations can be demonstrated are, first, through the 2-dimensional geometrical drawing of 3-dimensional solids, and, second, through the development of 2-dimensional nets of 3-dimensional solids. A net has been described by Borowski and Borwein (1991) as a diagram of a hollow solid consisting of the plane shapes of the faces so arranged that the cut-out diagram could be folded to form the solid. The first method of representation, the 2-dimensional geometric drawing, was explored by Mitchelmore (1980), and the second, students' ability to recognise and design nets, was investigated by Mariotti (1989) and Despina, Leikin and Silver (1999).

Mariotti (1989), in her attempt to identify specific didactic variables related to the utilization of mental images, hypothesised that there are two levels of complexity in the manipulation of mental images:

> - a first level when primary intuitions are sufficient: the image is global, it is not necessary to coordinate intermediate processes to solve the problem:
> - a second level when the primary intuitions are not sufficient any more: an operative organization of images is required to coordinate them according to the composition of transformation. (p.260)

In her study, she found that "constructing the correct net of the solid implies coordination of a comprehensive mental representation of the object with the analysis of the single components (faces, vertices and edges)" (p.263), and that it is possible to construct a hierarchy of difficulties in relation to the elaboration of mental images (p.265). Despina, Leikin and Silver (1999, p.4-248) in their examination of the problems related to finding different nets of a cube, found that students experienced great difficulty when attempting to work systematically.

While Mariotti (1989, p.263) argued that "constructing the correct net of a solid implies coordination of a comprehensive mental representation of the object with the analysis of the single components (faces, vertices and edges)", Despina, Leikin and Silver (1999, p.4-242), in investigating students' ability to transform the cube to different nets, found that "there is little information with respect to how students coordinate and analyze the components of a 3D solid when transforming it into a 2D net or vice versa."

Research design

The authors formulated three research questions related to Mariotti's study:

1. Can Mariotti's two levels of difficulty be demonstrated in students' descriptions of nets?
2. Do Mariotti's levels correspond with recognised frameworks of insight?
3. Is there a hierarchy of difficulties within Mariotti's second level?

To investigate these research questions and others, the authors developed a test, the responses to which could be analysed for information about students' understanding of 3-dimensional geometry. The questions covered the solid form and its cross-sections, as well as nets of 3-dimensional figures. This report considers the coding of the students' responses to one of the questions in which students expressed their understanding of a net. Two frameworks, the SOLO Taxonomy (Biggs & Collis 1982) and the van Hiele levels of understanding (van Hiele 1986) are used in the coding of the nature of thinking displayed by the students in their responses.

Background

The SOLO Taxonomy (Biggs & Collis 1982) has been identified by several writers (e.g., Pegg & Davey 1998) as having strong similarities with the van

Hiele Theory, despite having some philosophical differences. It is concerned with evaluating the quality of students' responses to various items. A SOLO classification involves two aspects. The first of these is a mode of functioning, and, the second, a level of quality of response within the targeted mode. Of relevance to this study are the three modes, ikonic, concrete symbolic, and formal. Within each of these modes, students may demonstrate a unistructural, multistructural or relational level of response.

In the *ikonic mode* students internalise outcomes in the form of images and can be said to have intuitive knowledge. By contrast, in the *concrete symbolic mode,* students are able to use or learn to use a symbol system such as a written language and number notation. This is the most common mode addressed in learning in the upper primary and secondary schools. When operating in the *formal mode*, the student is able to consider more abstract concepts, and to work in terms of 'theories'. A description of the levels that occur within the modes is given:

The *unistructural level* of response draws on only one concept or aspect from all those available.

The *multistructural level* of response is one that contains several relevant but independent concepts or aspects.

The *relational level* of response is one that relates concepts or aspects. The relevant concepts are woven together to form a coherent structure.

The targeting of the concrete symbolic mode for instruction in primary and secondary schools, and the implication that most students are capable of operating within the concrete symbolic mode (Collis 1988) has resulted in the exploration of the nature of student responses within the mode. This has led to the identification of at least two unistructural (U) - multistructural (M) - relational (R) cycles within the concrete symbolic mode (Pegg & Davey 1998). The most noticeable characteristic of the cyclic form of the levels is that the relational response (R1) in the first cycle becomes the unistructural element (U2) in the second cycle. This cyclical nomenclature has been used in the coding of concrete symbolic responses for this study.

Pierre van Hiele's (1986) work developed the theory involving five levels of insight. A brief description of the first four van Hiele levels, the ones commonly displayed by secondary students and most relevant to this study, is given for 3-dimensional geometry (Pegg's (1997) differentiation between Levels 2A and 2B is used):

Level 1 Perception is visual only. A solid is seen as a total entity and as a specific shape. Students are able to recognise solids and to distinguish between different solids. Properties play no explicit part in the recognition of the shape, even when referring to faces, edges or vertices.

Level 2A A solid is identified now by a single geometric property rather than by its overall shape. For example, a cube may be recognised by either its twelve equal edges or its six square faces.

Level 2B A solid is identified in terms of its individual elements or properties. These are seen as independent of one another. They include side length, angle size, and parallelism of faces.

Level 3 The significance of the properties is seen. Properties are ordered logically and relationships between the properties are recognised. Symmetry follows as a consequence. Simple proofs and informal deductions are justified. Families of solids can be classified.

Level 4 Logical reasoning is developed. Geometric proofs are constructed with meaning. Necessary and sufficient conditions in definitions are used with understanding, as are equivalent definitions for the same concept.

Van Hiele saw his levels as forming a hierarchy of growth. A student can only achieve understanding at a level if he/she has mastered the previous level(s). He also saw (i) the levels as discontinuous, i.e., students do not move through the levels smoothly, (ii) the need for a student to reach a 'crisis of thinking' before proceeding to a new level, and (iii) students at different levels speaking a 'different language' and having a different mental organisation.

Design and Initial Analysis of Responses

Among the questions in the 3-dimensional geometry test designed by the authors were six questions investigating students' understanding of nets. These questions probed for identification, description, explanation of relations and construction of nets. Initially, the test was given to over 1000 students from all except the final year (i.e., to Years 7 to 11) of four secondary schools in a rural city in New South Wales, Australia. This paper considers the responses to one of the questions on nets, namely:

"Describe in as much detail as possible what is a net of a solid."

Responses to the question were grouped according to the depth displayed in each answer. In an attempt to find some broad descriptors for the coding, the responses were initially grouped into three sections. Descriptions of each group, together with examples of responses matching the descriptors are given below:

Group A Responses

Responses in this group identified students who considered the net solely as a global image. There was no indication of awareness of a process of transformation between the solid and the net, or of properties or parts of the solid.

- what the solid is before it is a solid (A1)
- the 2D thing of a solid (A2)
- a solid flat (A3)
- a flat drawing of a solid (A4)

- is drawn like below (diagram) (A5)
- the outside of a solid (A6)

Group B Responses

In these responses, students showed awareness of the process of conversion from the 2-D to the 3-D image and/or vice versa, by folding/unfolding, putting together/taking apart, or by spreading out flat. However, there was no indication of awareness of the properties or parts of either the solid shape or the net.

- the unfolded 3 dimensional shape (B1)
- something that can fold in to make an object, e.g., this is a pyramid net of a solid (diagram) (B2)
- the solid taken apart and flat (B3)
- a shape spread out ... when you fold it back together it forms that shape (B4)
- the combination of 2D shapes that you need to form a 3D shape (B5)
- the pattern made before it is put together ... the outline of the solid (B6)
- the shape and parts of an enclosed space, when it is taken apart and laid flat to make a shape (B7)
- a plan of a solid that has been opened (B8)
- what I have drawn on the right (diagram) ... this net makes a triangular prism (B9)

Group C Responses

Responses in this group contrasted to those in group B in that in addition to the process of transformation between a solid and a net, the students showed in their answer that they were aware of at least one part (faces, edges and vertices) of a solid. Included in these responses were those which indicated the need for edged to correspond.

- a description of the solid with its faces, edges, and vertices in a flat layered out position ... it shows the different faces on sections of the shape (C1)
- the shape and all its faces folded out (C2)
- a solid kind of flattened out ... a plane shape with sides on it (C3)
- the plan (like a building plan) that shows all the sides in one picture (C4)
- a 3 dimensional shape flattened out to make a 2 dimensional shape ... the edges are separated to make it flat (C5)
- the faces laid out flat ... you can see only the framework for a shape (C6)
- the outline of the skeleton of a solid ... it is a 2D depiction of the solid as if it were unfolded ... from the net, each line represents an edge and each area represents a face ... when make into a 3D solid, the vertices form as a result of joining the edges and faces together (C7)
- a 2-D representation of a solid ... when all sides are joined together it must make a solid, so these 2-D shapes must be in the correct position on the net

> ... the sides of the 2-D net must be correct lengths so that when they are joined together there must be no uneven or left over edges (C8)

It is considered that the above grouping supports the first research question, i.e., they demonstrate Mariotti's two levels of difficulty. Group A corresponds with the first level in that the responses indicate solely a global perception of a net, whereas, groups B and C, in demonstrating awareness of the process of transformation between the 3-dimensional and 2-dimensional shapes, correspond with her second level.

Final Coding

The responses within the above initial groups were coded with relation to the SOLO Taxonomy. The first consideration was to classify the responses according to their targeted mode of functioning. Group A responses were considered to illustrate students who were perceiving a net intuitively as a visual image. Hence, these responses were classified as being of the ikonic mode. The responses in the latter two groups, B and C, were both considered to be perceiving the net as a symbol or representation of the solid, with group C perceiving the net additionally as corresponding to the parts of the solid. Responses belonging to both these groups were classified as being of the concrete symbolic mode. No responses were considered to display the formal mode of functioning.

The nature of the responses for group A were not considered further. However, groups B and C responses, both deemed as concrete symbolic in nature, were further classified for the cyclical levels within the mode. It was considered that responses in group B, those displaying awareness of transformation between the 3-dimensional solids and 2-dimensional nets, but not identifying any correspondence between the parts of the solid and the shape of the net, were in the first cycle. These responses were coded for the level demonstrated according to whether they indicated a single process in the transformation, e.g., unfolded/flattened, or whether they considered the transformation as a multiple process, i.e., those responses indicated either that the process was reversible, or that it consisted of two or more actions, e.g., cut and unfolded. Responses of the first type were coded as being unistructural in the first cycle (U1). In the examples, it is considered the responses B1, B5, B8, and B9 are unistructural, first cycle. Those of the second type were coded as being multistructural in the first cycle (M1). Responses coded as M2 from the examples are B2, B3, B4, B6, and B7. R1, or relational responses of the first cycle become the unistructural element (U2) in the second cycle.

Responses which identified part(s) or property(ies) of a solid were deemed to belong to the second cycle of levels. Identification of a single part or property indicated a response as being unistructural (U2), for example, responses C2, C3, C4, C5, and C6. Reference to more than one property or part of the solid classified a response as multistructural (M2), for example, response C1. If a response also included information concerning a correspondence between the faces or edges of the solid and the sections of the net, or of the matching of edges in the reconstruction of the solid from the net, the response was coded as relational (R2). Responses which matched this coding are C7 and C8.

Finally, the codings were related to the van Hiele descriptors. It was considered that the ikonic responses indicated a visual understanding, corresponding to van Hiele Level 1. Responses which demonstrated an appreciation of the physical process involved in the transformation between a solid and its net only were considered not to show awareness of properties, yet were exhibiting more than visual awareness. Hence, these responses have been coded as transition, i.e., between van Hiele Levels 1 and 2A. Responses which indicated awareness of a single property or part of a solid, and were coded as U2 were considered to demonstrate van Hiele Level 2A understanding, while responses which were similar but showed also awareness of more than one property or part of a solid were considered to demonstrate van Hiele Level 2B understanding. Finally, it was considered that responses which included a reference to the relationship between the faces and/or edges of the solid and the net demonstrated understanding at van Hiele Level 3.

A summary of the coding of the nature of thinking displayed in responses to the question 'Describe in as much detail as possible what is a net of a solid' is given in Table 1.

Table 1
Coding of the nature of thinking displayed in responses on nets of solids

Description	SOLO mode/level	van Hiele Level
visual only		
a flattened shape	Ikonic	1
perceiving the process		
a shape folded/unfolded	CS/U1	transition
the above plus a linkage (cut, take apart, make up)	CS/M1	transition
identifying part or parts of the net	CS/R1	transition
considering the shapes		
edges/faces matching or some property of the net	CS/U2	2A
combination of the above	CS/M2	2B
relational position of the faces	CS/R2	3

The coding of the responses using the two frameworks, the SOLO Taxonomy and the van Hiele levels of understanding, not only confirms the first research question, that Mariotti's two levels of difficulty can be identified in students' descriptions of nets, but also supports the second and third research questions. Mariotti's first and second levels correspond respectively with the ikonic and concrete symbolic modes in the SOLO Taxonomy, and also with van Hiele Level 1, and van Hiele Levels 2 and 3. Further, the coding within the two frameworks clearly demonstrates that Mariotti's second level can be partitioned into a hierarchy of degrees of insight. This is shown in the two cycles of levels in the SOLO Taxonomy and in the correspondence between Mariotti's second level and van Hiele's Levels 2 and 3.

Conclusion

In this study, the authors have investigated further the insight required in visual reasoning in relation to nets of 3-dimensional solids. Three research questions developed from Mariotti's study have been demonstrated to hold true. This now provides the theoretical evidence necessary to explore the issues further.

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A comparison between Malaysian and United Kingdom teachers' and students' images of mathematics

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Abstract:

This paper discusses the findings of a cross-cultural study that aims to explore and compare images of mathematics of 406 Malaysian teachers and students with that of 231 United Kingdom (UK) teachers and students. A short questionnaire consisted of two open-ended questions and nine structured questions were used. The overall findings indicate that there are cultural differences in teachers' and students' images of mathematics between the two countries. In particular, the Malaysian teachers and students viewed mathematics as an enjoyable and challenging subject. As they attributed success in mathematics to effort and persistence, they viewed learning mathematics as an effortful endeavour. However, most UK teachers and students attributed success in mathematics to inherited mathematical ability. Many believed they did not have this ability and consequently many of them related mathematics to boredom and difficulty.

Introduction

For the last three decades, there have been an increasing number of cross-cultural studies focusing on comparing mathematics performance of students between Western industrialised countries such as USA, Eastern Europe and UK, and Asian countries such as Japan, China, Taiwan and Korea (e.g., Husen, 1967; Song & Ginsburg, 1987; Stigler & Perry, 1988; Robitaille & Garden, 1989; Stevenson & Lee, 1990; Stevenson & Stigler, 1992; Cai, 1995; Beaton, Mullis, Marton, Gonzalez, Kelly, & Smith, 1996; Keys et al.,1997). Generally, these studies display that most Asian students have outperformed most of their counterparts from United States and United Kingdom. There is also an increasing number of research studies on cultural beliefs and values, as well as on social and parental expectations which seek to explain these differences in mathematics performance and motivation in learning mathematics. (See example, Ryckman and Mizokawa, 1988; Stevenson, Chen, & Uttal, 1990; Samimy, Liu, & Matsuta, 1994; Eaton & Dembo, 1997; Gipps & Tunstall, 1998; Huang and Waxman, 1997; Elliot, Hufton, Hildreth & Illushin, 1999).

These studies have consistently displayed that there are cultural differences in mathematics achievement as well as on attitudes towards mathematics. The differences in societal beliefs and cultural values have been cited as one of the possible explanations. The most popular one being Eastern society values effort, perseverance and hard work while western family tend to rely on mathematical ability and creativity as important contributing factors for success in mathematics (e.g. Ryckman and Mizokawa, 1988; Huang and Waxman, 1997). Albeit inconclusive, the latter claim has been challenged by more recent cross-cultural studies (e.g. Gipps & Tunstall, 1998; Elliot et al., 1999). Gipps and Tunstall (1998) asked 49 English young children (6 and 7 years old) to give reasons for success or failure related to some short 'stories' about classroom performance. Their findings show that effort was the most commonly cited reason, instead of competence in the specific subject areas.

Mathematics education by itself is a socially constructed activity. Teachers and students construct different images towards 'mathematics' because they have different experiences and different level of understandings of mathematics. Likewise, people from different cultural background may view mathematics education differently. As yet, there is little cross-cultural comparison study on images of mathematics. Thus, a comparison between two countries on their teachers' and students' images of mathematics may provide us with an insight into the possible influence of cultural values on images of mathematics. Malaysia and United Kingdom (UK) have been closely linked to each other historically (as one of the colonial country), particularly in terms of education system and mathematics curriculum. However, differences in language, cultural practices, beliefs and value system may impact in significantly different ways on their people. Thus, the findings of this study might serve to support or challenge the recent propositions that images of mathematics are culture-bound and value-laden as proposed in some literature (see e.g., Bishop, 1988; Ernest, 1991).

Definition of 'image of mathematics' in this study

Reviewed of past literature indicate that there is still a lack of consensus on the definition of 'image of mathematics'. This term has been used loosely and interchangeably with many other terms such as conceptions, views, attitudes and beliefs about mathematics. However, in this study, I chose to adopt both Thompson's (1996a) and Rogers' (1992) suggestions and define the term *image* as some kind of mental representation (not necessarily visual) of something, originated from past experience as well as associated beliefs, attitudes and conceptions. Thus the term *'image of mathematics'* is conceptualised as a mental representation or view of mathematics, presumably constructed as a result of social experiences, mediated through school, parents, peers or mass media. This term is also understood broadly to include all visual and verbal representations, metaphorical images and associations, beliefs, attitudes and feelings related to mathematics and mathematics learning experiences.

As discussed elsewhere (see Lim & Ernest, in press), the term 'image of mathematics' is conceptualised to include the following components:

1. Stated attitudes (reported liking and disliking mathematics)
2. Feelings (choice of emotive descriptors)
3. Description/metaphor for mathematics
4. Beliefs about the nature of mathematics
5. Views about mathematicians and their activities
6. Beliefs about mathematicians' ways of knowing and warranty of mathematical knowledge
7. Description/metaphor for learning mathematics
8. Aims for school mathematics
9. Beliefs about mathematical ability
10. Beliefs about gender differences in mathematical ability

Subsequently, these components were used in designing the questionnaire for the study.

Respondents

A total of 100 teachers and 306 students from Malaysia while 67 teachers and 164 students from UK responded to the questionnaires. Table 1 displays the frequency distribution of the respondents in terms of gender and occupational groupings. The Malaysian teacher respondents were from two teacher-training colleges where these teachers were attending an in-service course. All these teachers were qualified primary school teachers. 36 of them were mathematics specialists while 64 of them specialised in English or science. Similarly, the Malaysian student respondents were collected from the same teacher-training colleges. These students were attending a two-year full-time course in teacher education. 173 of them were majoring in either primary mathematics or secondary mathematics while 133 were majoring in English, science or other general subjects.

The UK teacher respondents were mainly from various schools around South West of England. In particular, 12 of the mathematics teacher questionnaires were collected during a mathematics teacher seminar. All the mathematics and non-mathematics student respondents were students of the University of Exeter. Their majoring subjects varied. They ranged from social science, education to technology.

Table 1: Frequency distribution of the Malaysian (and UK respondents)

Gender	Occupational Grouping				Total
	Maths teachers	Non-maths teachers	Maths students	Non-maths students	
Male	15 (12)	27 (12)	55 (22)	38 (53)	135 (99)
Female	21 (17)	37 (26)	118 (25)	95 (64)	271 (132)
Total	36 (29)	64 (38)	173 (47)	133 (117)	406 (231)

I acknowledge that as these respondents were collected by opportunity and thus not representative, and all results and findings discussed in this paper are thus limited and not generalisable to include the wider population of students and teachers of these two countries.

Method

As this study aims to explore and compare teachers and students' images of mathematics between two countries, I opted for survey method using questionnaire to collect the data. However, I was aware of the limitations of survey methods. Kerlinger (1986) reminds us that "survey information ordinarily does not penetrate very deeply below the surface. The scope of the information sought is usually emphasized at the expense of depth" (p.387). In view of this limitation, instead of the usual five-point Likert scales, I have structured the questionnaire with a variety of question types, including open-ended question, structured questions with 'yes-unsure-no', 'ranking in order of importance' and choosing as many options described from a large selection.

Moreover, images are personal constructs, and these are often subtle and complex. Thus, it is difficult to use any simple language to frame the questions in such a way that it will suit all respondents' view. More importantly, I wanted to understand the images of mathematics from the respondent's point of view, rather than comparing them with certain fixed or existing pattern of images. As proposed by Mura (1992) that the use of open-ended questions "is able to capture some spontaneous thoughts of the respondents which might be the most salient and this format allows the respondents to express mixed perceptions" (p.4). Therefore, I decided to use an open-ended question approach, which allows the respondent to respond more freely, even though I recognised that the responses will be varied and consequently could be more difficult to be interpreted and analysed.

A short questionnaire was designed to collect the data. It consisted of two open-ended questions and 9 structured questions. The open-ended questions (Question 2) asked for participants' images of mathematics and learning mathematics in the form of descriptions, and metaphors or analogies.

Initially I hesitated as to whether I should give an example for each of these open-ended questions, as I understand that any example given might bias the responses obtained. However, during piloting, I was asked for an example of an image by a number of respondents. Thus, I decided to give an example of metaphor to act as a trigger as well as to clarify the question. I acknowledge that whatever type of example that I give (such as positive or negative image) might influence the type of answer that I obtain. So I chose to give an example of an image in the form of metaphor, and which was neutral with regard to positive or negative images of mathematics.

The nine structured questions were intended as a mean of investigating the following constructs:

I. attitudes - Question 1 (liking) and Question 3 (feelings, self-confidence and anxiety about mathematics);

II. beliefs - Question 5 (belief about mathematical ability); Question 6 (belief about gender difference in mathematical ability);Question 7 (belief about the nature of mathematics) and Question 8 (belief about the importance of school mathematics);

III. images - Question 9 (images of a typical mathematician) and Question 10 (images of how a mathematician finds new knowledge).

Some modifications on the questionnaire were made to suit the need of the Malaysian respondents. Firstly, the questionnaire was translated into Bahasa Malaysia, the official language of Malaysia. This is also the medium of instruction in all Malaysian government-owned educational institutes. The questionnaire was first translated by myself and then validated by two Malaysian research students (specialising in English language) to ensure that the translated copy was as equivalent as possible in content and meaning to the English version.

For the open-ended question on images of mathematics and learning mathematics, all responses of the Malaysian respondents (except 53 of them in English) were given in Bahasa Malaysia. I first translated all these responses into an English version before categorising them. The examples given in the following sections are verbatim responses that have been translated into English version. I used the notation, 'M000' to denote response of the Malaysian respondents while 'R000' for response of the UK respondents (where the three zeroes indicate where 3-digit response numbers are inserted).

For brevity, from now on, I shall refer the teacher and student respondents from Malaysia as 'the Malaysian teachers and students'. Similarly, the UK respondents of teacher and student will be referred as 'the UK teachers and students'. However, in view of the nature of sampling discussed earlier, they are by no mean representative of all teachers and students from both countries.

Findings and discussion

The findings of this cross-cultural study show that overall, there were more differences than similarities in images of mathematics among the teachers and students between the two countries. Some striking differences were found in terms of reported liking (disliking), images of mathematics and learning mathematics; beliefs about the nature of mathematics, beliefs about attribution factors to mathematical ability, and images of mathematicians and their works.

First of all, the Malaysian teachers and students displayed relatively more positive attitudes towards mathematics than the UK teachers and students. They showed higher percentages in reported liking (81.6% as compared to 58.9%) and relatively lower percentages in reported disliking of mathematics (only 5.7% instead of 31.2%). In particular, the Malaysian non-mathematics students displayed noticeably higher percentage of reported liking than their UK counterparts (69% instead of 37%). While 49.6% of the UK non-mathematics students (particularly the age group 17-20 years old) reported disliking mathematics, only 9% of Malaysian non-mathematics students reported similarly.

Secondly, in terms of images of mathematics, the Malaysian teachers and students commonly viewed mathematics as a subject that was enjoyable and challenging, mainly made up of number and symbol or involved mental thinking. Some examples of image of mathematics commonly given were:

> an enjoyable and challenging subject (M134, non-maths student, female, 21-30);
>
> an activity that involved numbers and precise calculations (M220, maths teacher, male, 21-30)
>
> improve your mind to think precisely and accurately (M293, non-maths student, male, 21-30)

In contrast, more than 10% of the UK teachers and students expressed their images of mathematics as related to boredom and difficulty. For example, mathematics is,

> really difficult and confused - like getting lost (R397, maths student, female, 17-20)
>
> boring, hard & very academic (R512, maths student, male, 17-20)
>
> difficult, unimaginative and dull (R64, history student, male, 17-20)
>
> difficult subject (R477, drama teacher, male, 31-50)

This is particularly notable among the non-mathematics students aged between 17-20 years old of the UK respondents. Perhaps this is not a surprise in view that this group was also the group who reported disliking mathematics the most.

In terms of images of learning mathematics, the Malaysian teachers and students more often viewed learning mathematics as an activity that was an effortful endeavour such that:

> like eating durian - need to work hard to open it (M166, maths student, male, 21-30) [N.B. 'durian' is a Malaysian fruit with a thorny and hard shell)
>
> like climbing up the mountain peak because you need to work hard, patient and persistence (M006, maths student, female, 17-20)
>
> doing a job that need intense observation and hard work (M093, non-maths student, male, 21-30)

These metaphors stress the need for continual work, and the importance of efforts, hard work and persistence in learning mathematics. Although both the Malaysian mathematics teachers and non-mathematics teachers noted 'difficulty' in their images of learning mathematics, the former seemed to perceive it as a process that is 'difficult but rewarding'. For example, the Malaysian mathematics teachers perceived learning mathematics as,

> mountain climbing - the journey is difficult but reach the top at the end (M224, primary maths teacher, female, 31-50)
>
> a challenge that need to be faced with, satisfying when successful (M236, primary maths teacher, female, 31-50)

These images of learning mathematics suggest that the Malaysian mathematics teachers also found mathematics difficult but they took it as a challenge and something they must work towards that will bring success eventually. Interestingly, these views were shared by over 5% of the Malaysian mathematics and non-mathematics students too.

In contrast, images of learning mathematics given by the UK teachers and students were more varied and diverse. There was no common image of learning mathematics that was shared by all the four subgroups as compared to the Malaysian respondents. The results show that learning mathematics was equated to 'problem solving', learning a 'skill' or 'games or puzzles' by nearly 5% of the UK teachers and students. Although there was a typical example of images of learning mathematics given by an UK non-mathematics student:

> banging my head against a brick wall (R054, non-math student, female, 21-30)

Interestingly, six other UK non-mathematics students also gave this same simile.

In comparing the mathematics teachers from both countries, it is interesting to observe that they shared a rather similar image of learning mathematics. One of the Malaysian mathematics teachers gave his image of learning mathematics as,

> like fishing - once you have a catch, you want to do it again and again (M228, primary maths teacher, male, 31-50)

This view of learning mathematics as related to something difficult but rewarding was also shared by one of the UK mathematics teacher:

> like climbing the mountain - worth the view (R292, maths teacher, female, 31-50)

Likewise, many non-mathematics teachers from both countries perceived the learning of mathematics negatively. While the Malaysian non-mathematics teachers tended to view learning mathematics as,

> cracking your head, watching your brain melting down (M356, primary non-maths teacher, male, 31-50)

The UK non-mathematics teacher viewed learning mathematics as:

> walking through mud (R155, non-maths teacher, female, 31-50)

Again, all these metaphors indicated negative feelings of 'difficulty' and 'struggling'. While over 10% of the Malaysian non-mathematics teachers still viewed learning mathematics as enjoyable, only two UK non-mathematics teachers (5%) shared this positive feeling. Nevertheless, the number of respondents is too small to infer too much from these results.

Thirdly, the results suggest that there is a cultural difference in the factors to which a person's mathematical ability is attributed. While the majority of the Malaysian teachers and students attributed 'effort and perseverance' as the most important factor, the majority of the UK teachers and students ranked 'inherited mathematical ability' as the most important factor. This finding concurs with that of many cross-cultural studies ((Ryckman and Mizokawa, 1988; Stevenson & Stigler, 1992; Huang and Waxman, 1997) that Eastern families tend to value effort, perseverance and hard work whereas Western families tend to rely on mathematical ability and creativity as important contributing factors for success in mathematics.

On the other hand, this finding seems to challenge the findings of two recent studies (Gipps & Tunstall, 1998; Elliot, Hufton, Hildreth & Illushin, 1999), which observed that some English young children had quoted effort rather than ability as the major reason for success in mathematics (Gipps & Tunstall, 1998) and in school work (Elliot et al., 1999). Perhaps this is an optimistic notion and that there might have been a shift in belief about mathematical success in the younger English generation. However, samples in these studies, including those in this cross-cultural study, are by no mean representative of their countries, thus these findings remain inconclusive unless more consistent evidences are found.

Overall, the above two findings seem to suggest that the images of mathematics and learning mathematics were closely linked to beliefs about mathematical ability. As the Malaysian respondents believed that effort and perseverance were important factors for success in mathematics, they viewed learning mathematics as an effortful activity. They admitted that learning mathematics could be difficult but they perceived the difficulty as a challenge, not as an obstacle. This belief of ability as driven by effort and the image of mathematics as a challenge might promote a willingness to work harder. This respondents might then have achieved a sense of satisfaction and enjoyment at their success. As a result, they might tend to display a more positive attitude towards mathematics and learning mathematics. This is a plausible chain of causal links, but without further confirmation, it remains a speculation.

In contrast, the UK respondents attributed success in mathematics to 'inherited mathematical ability'. Consequently, they might not believe that working hard would help them to achieve better, especially if they might believe that they themselves lack mathematical ability. They might tend to believe that mathematical success was only possible for a few clever ones. They might tend to give up more easily and to see themselves as failures in mathematics. As a result, they might not accept difficulty in mathematics as a challenge. Instead, they might tend to adopt a negative attitude towards mathematics and easily feel bored and frustrated, even when they have experienced little failure in mathematics. Again, this is a plausible chain of causal links, but without further confirmation, it remains a speculation.

Lastly, while the Malaysian teachers and students tended to adopt an absolutist view of mathematical knowledge, the UK teachers and students tended to subscribe to a more fallibilist view. For the majority of the Malaysian teachers and students, mathematical knowledge was exact and certain. There was always a right answer to any mathematical problem, even though some of them believed that there were more than one way to solve a mathematical problem. They also believed that mathematicians tended to use fixed and rigorous methods such as 'testing' and 'complicated calculation', and formulation. Methods such as 'guessing', 'intuition' and 'falsification' were considered as 'improper' methods.

In contrast, the UK teachers and students displayed a wider range of views about mathematical knowledge. The mathematics teachers tended to exhibit a relatively stronger inclination towards a fallibilist view of mathematical knowledge whereas the students and the non-mathematics teachers exhibited a mixture of absolutist and fallibilist view of mathematical knowledge.

Conclusion

While it should be recognised that the two samples studied are by no mean representative of the two countries, the findings do serve to highlight the possible influence of culture and values in the formation of images of mathematics among some teachers and students of both countries. The comparison made in this cross-cultural study both supports and challenges the notion that image of mathematics is value-laden and culture-bound. By and large, cultural differences in beliefs about mathematical ability and the importance of mathematics education might have resulted in different emphases in the process of learning mathematics in school. As a result, differences in mathematics learning experiences might have brought out differences in people's images of mathematics and learning mathematics. Consequently, this might conceivably have given rise to the differences in image of mathematics and learning mathematics as well.

On the other hand, it might be argued that the differences could be due to the different influences of significant others in the two countries (such as mathematics teachers and parents), or the different cultural presuppositions. As Elliot, Hufton, Hildreth & Illushin, (1999) point out that "perhaps more important are children's familial, peer and cultural perceptions about what constitutes real and meaningful educational achievement and the extent to which this is seen to be of such intrinsic or extrinsic values as to evoke significant effort" (p.91). Thus, I acknowledge that due to the limits of inferences that can legitimately be drawn from a questionnaire survey, these claims remain speculative and need to be confirmed by further research studies on these matters.

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ON DEVELOPING TEACHERS KNOWLEDGE BY USING CASES CONSTRUCTED BY RESEARCHER AND CLASSROOM TEACHERS

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ABSTRACT

This study was designed to enhance teachers' knowledge by constructing cases of teaching as part of a school-based professional development project. Cases were developed collaboratively by a school-based team consisting of the researcher and four classroom teachers. The participants' conceptions of cases and their skills for cases writing were developed in the study. By constructing cases of teaching, participants increased their pedagogical content knowledge and became more reflective practitioners.

Introduction

The reform of teacher education supports a closer examination of alternative methods of instruction in teacher education programs (NCTM, 1991). The conviction has been growing that the use of cases or narratives may be more helpful for those who need to learn to think in new ways about complex problems of teaching (Carter, 1993; Harrington, 1995; Richardson, 1993). More recently, narrative case materials have become available that address dilemmas in teaching elementary mathematics (Schifter, 1996; Silver, 1999). Some research shows that even though teachers are exposed to theories of learning and teaching in some in-service teacher education programs, they do not seem able to apply this knowledge in actual classroom practice (Schifter, 1996, Richardson, 1993). The use of cases is a way to help illustrate the critical processes as teachers try to translate their theories into classroom practice (Barnett, 1991; Carter, 1993; Richardson, 1993), since learning to teach develops in part by focusing on understanding the dilemmas of teaching (Harrtington, 1995).

Understanding the dilemmas of teaching is similar to overcoming the contradiction of cognition. Contradiction produces a refining of one's cognitive structure (Perret-Clermont, 1980). Conflicts of cognition may result from two concepts that both seem plausible and yet are contradictory, or concepts that become insufficient given new evidence. In each of these cases, contradiction causes an imbalance providing the internal motivation for an accommodation. Piaget (1971) notes that the contradictions are constructed only secondarily, after learners first search for similarities between experiences and attempt to organize each experience with their present schemes. He asserts that the way one accommodates cognitive structures which are in disequilibrium is to modify and to reorganize one's current schemes, and thus cognitive development is achieved. With this in mind, this study was designed to create opportunities for teachers to develop specific and deeper mathematical and pedagogical content understanding through observation and discussion.

Cases as exemplars can be used not only to provide opportunities for exploring the complex problems of teaching but also for stimulating personal reflection in teaching (Merseth, 1992; 1996). The cases referred to in the study contained three elements as defined by Merseth (1996): The cases were real situation based on teaching; the cases were constructed by the study, and the cases provided statements or data for consideration and discussion by users. The use of cases includes both case discussion and case writing. Case discussion can play a critical role in expanding and deepening pedagogical content knowledge (Barnett, 1991). The case writing process itself involves an internal process, while case discussion fosters personal reflection through an external process (Shulman & Colbert, 1989). Although some research on the influences of case discussion on what and how teachers think has been carried out (Richardson, 1993; Merseth, 1996), limited research has been conducted on the development of case writing (Lin & Tsai, 1999). In Shulman's study, case writing is developed by taking a teacher's self-report and turning it into a teaching case (1992). There is very little research showing the effects of cases on teachers' knowledge when cases are constructed by a collaborative team consisting of a university professor and teachers participating in a school-based teachers' professional development project.

The professional development project upon which this research paper is based was built on the social constructivists' view of knowledge, in which knowledge is the product of social interaction via communication and dialects in a community (Vygotsky, 1978). Therefore, activities were structured to ensure that knowledge was actively developed by teachers, not imposed by the researcher. The cases in the study were developed by focusing on the dilemmas of teaching engendered by participants' professional dialogues and by providing them with opportunities to examine their teaching practices. Participants were frequently involved in observing teaching, dialoguing as a group, and reflecting on what they had observed.

The study reported in this paper was designed to enhance teachers' knowledge by constructing cases of teaching as part of a school-based professional development project. There are two research questions: 1) What are teachers' conceptions of cases? 2) How does the use of cases influence teachers' knowledge? Teachers' knowledge referred to in this study included the ways of representing and formulating mathematics concepts that make them comprehensible to students. Also included in teachers' knowledge is the understanding of how students think and what strategies students use to solve problems.

Methodology

It's not possible to adequately illustrate the process of teachers' development without dialoguing about critical pedagogical issues in which they participate (Cobb and McClain, 1999). This indicates that there is a need for what teachers are learning in their own classrooms to be communicated to their colleagues. Thus, a school-based collaborative team was set up in this study to discuss the situations which occurred in one teacher's classrooms by comparing them to others. Classroom observation was used as a means of achieving the goal.

Ding-Pu, with about 780 students and 36 staff, is located in an urban area of Taiwan. The size of each class in the school is around 35 students. The school was selected because of teachers' willingness to learn, principal and administrators' support, and near to teachers college where the researcher teaches. The participants consisted of a collaborative team including the researcher and four first-grade teachers, who had taught between eight and fourteen years.

Participants with distinct experiences and beliefs were asked to share and discuss the issues of pedagogy and students' learning in regularly weekly meetings lasting for three hours on every Monday afternoon. The researcher was expected to contribute more theory than practice, while the four teachers were expected to share more classroom experiences in regular meetings. The researcher was a partner of the teachers in helping them put the ideas generated from discussions into practice. The first-grade classrooms were the primary contexts for participants to frame problems, analyzing situations, and argue the advantage and disadvantage of various alternatives. There were two reasons for selecting teachers from the same grade to participate in this study. One reason was that the participants confronted similar pedagogical problems when they taught the same lessons. The same mathematical content easily lent itself to a focus when participants met together to address issues and solve pedagogical problems after observing others' lessons. The second reason was that similar pedagogical issues addressed in the regular meetings drew attention and concern from each participant, and led to in-depth discussions.

Three phases of constructing cases were considered for developing teachers' conceptions about cases and enhancing their professional knowledge and reflections on classroom practice. The first phase was to probe for teachers' responses to cases written by the researcher. The cases of teaching were developed from several observations by first discussing the issues addressed in participants' lessons, and then writing them into a complete case form. The cases used in the first phase were developed from the participants' concerns about how to present mathematical contents and about students' learning. The second phase was to assist teachers in constructing their own teaching cases. To developing teachers' conceptions of cases, three questions were discussed in a regular meeting: (1) What are the distinctions among the cases provided? (2) What content might be included in a case? (3) How can one construct a teaching case? The purpose of the third phase of constructing cases was to develop teachers' ability in writing cases. The process of constructing a complete case with a narrative form involved first observing and dialoguing professionally about a lesson to initiate a case. Then accounts of teaching processes and illustration of the kinds of dilemmas which arose in instruction were discussed in regular meetings, to elicit and revise the case.

The data collected for this study included classroom observations and all discussions for constructing cases in regular meetings. In addition, each teacher was interviewed individually to collect his or her responses to the processes of conducting cases. The changes in participants' responses over time were an indication that their pedagogical knowledge had been enhanced. The interviews and group discussions were first audio-taped and then transcribed verbatim. The transcriptions were coded and then laid out on 8k size paper, sentence by sentence, and were clustered for each participant.

Results

There were two main themes which emerged. The first theme, answering the first research question, is relevant to teachers' conceptions of teaching cases. The second theme, answering the second research question, shows the influence of cases on teachers' professional knowledge and teachers' becoming more reflective practitioners.

Theme 1: Teachers' conceptions of teaching cases: Types and contents covered in cases and ways of constructing cases

Participants distinguished between three types of teaching cases. Teacher's analysis of children's strategies is the first type of case. Students' interpretations of other students' thinking is the second type of case. The purpose of this type of case is to expand teachers' view of students' critical thinking ability. The third type of case shows two approaches to a specific topic, which increases teachers' awareness of alternative approaches.

Participants noted that it was important that the content included in cases to be described clearly and concisely; particularly, the goal of instruction, students' prior experiences required, and the context of teaching were three essential elements that had to be stated clearly in a case. The text of a case varies with the type of cases. For instance, text belonging to the third type of case needs to portray the distinctions between the two approaches. To establish a case, participants suggested that the following procedure: 1) reading the teachers' guide and goals of instruction; 2) performing the teaching, and 3) observing one's own teaching carefully.

Theme 2: (1) Influences of using cases on teachers' knowledge: Teachers increased pedagogical content knowledge

The narrative cases developed from participants' observations helped teachers categorize the semantic structures of addition and subtraction word problems. In Sue's lesson, the goal was to develop students' ability to solve subtraction word problems by using chips. Three problems described below were proposed by Sue in the lesson.

Problem 1: There are 3 red and 8 green chips. How many fewer red chips are there than green chips?
Problem 2: There are 6 baseballs in the playground. Steven took 2 of them away. How many baseballs are there now?
Problem 3: There are 8 boys playing in the sand and 3 girls playing on swings. How many more boys are there than girls?

Participants were asked to distinguish between the three problems and categorize students' strategies of solving them. Through discussing, teachers recognized the distinctions among "take away" problems, compared problems with the relational term "more than", and compared problems with the relational term "less than". In addition, teachers' awareness of levels of problems was developed. They appeared to have more concerns about examining the sequence of curricular material based on students' learning, as revealed by Yo, one of the participants, below:

> "The cases we developed not only help teachers to understand the ways children think but can also help us examine whether the flow of teaching and the organization of curriculum matches with children's cognition. (Yo, group discussion, 981204)"

At the end of the first half term of the study, each participant was to investigate the first graders' difficulty with mathematical problems using the relational term "less than". By observing their teaching, it was found that the pattern of erroneous mathematical expression for first-grade students was $8 - 13 = 5$. We discussed the text of the case being developed, including three participants' problems posed in their own classes and students' various mathematical expressions for the problems. The problems described in the case included:

(1) There are 13 umbrellas and 8 students. Which number is fewer? What is the difference between the two numbers?
(2) Joe has 13 marbles. Tom has 8 marbles. How many fewer marbles does Tom have than Joe?
(3) David has 8 dollars and Kris has 13 dollars. How many fewer dollars does David have than Kris?

The participants were asked to make analyses on the higher percentage (45%) of students who used the wrong mathematical expressions 8 - 13 = 5 for problem (3) compared to the other problems (3%, 6%). They noted that in problem (3) the smaller number (8) was presented in the text of the problem before the larger number (13). Therefore, students who were not acquainted with the meaning of the mathematical expression could not solve problem (3) using a one-to-one correspondence strategy successfully. They realized that there is one more procedure to solving problem (3) than problem (1) and (2), if first graders are to answer these problems successfully. The additional procedure required for problem (3) is to transpose the question sentence "How many fewer dollars does David have than Kris?" to "How many more dollars does Kris have than David?". As a consequence, teachers understood that this type of problem is not appropriated for first graders who have not developed the level of reciprocal relation between subtraction and addition.

In addition, participants supported the significance and value of teaching cases for developing their deep understanding of methods to motivate students. After using cases, teachers' concerns during classroom observations shifted from teachers' behaviors and classroom environment to students' learning. Cases that were developed from participants' observations and written by the researcher helped teachers explore children's development of mathematics ideas more in-depth. The following is an excerpt from group discussions.

> "When observing a lesson, we only observed on the surface without grasping children's thinking processes. The strategies listed in the cases make me explore deeper about children's development of mathematics ideas. I am able to know those who understood or misunderstood and to know the ways of children's thinking. (Huei, group discussion, 981127)"

Theme 2: (2) Influences of using cases on teachers' knowledge: Teachers became more reflective practitioners

Having teachers respond to a case seemed to reinforce further insights into their thinking with respect to mathematical learning. Teachers became active contributors of multiple perspectives and became more reflective about their classroom practices. In addition, the third phase of constructing cases designed to develop teachers' ability in writing cases, was likely to stimulate personal reflection. In the first meeting for conducting the case, participants' lessons and the ensuing discussions were the particular focus or frame for problems. The discussions based on the participants' perspectives reported in the meeting, were organized to be the content of the first version of the case.

The preparation of cases seemed to help teachers develop skill central to reflective practice, for instance, learning to focus on alternative approaches. An example was when Ling prepared the first version of a case conducted from her teaching. The goal of Ling's lesson was for students to learn to count by tens to 100. When first observed, Ling paid more attention to the differences of students' performance compared to Jong's class. Ling perceived that the differences in strategies used by students in the two classes were because of their different methods of teaching. Thus, Ling used the comparisons of two instructional approaches as the main text of the case to be built. However, Ling didn't describe the key element in her teaching, which resulted in her students employing multiple strategies, until Jong addressed the main difference of their two approaches: Jong's students were asked to bind ten straws into a bundle, while Ling's students were not. Thus, the case-development process internalized Ling's insight into how to improve her teaching students to count by tens. It is evident that writing using the context of Ling's personal teaching enhanced her and other participants' reflection on their teaching. Moreover, the case Ling constructed effectively included her self-reports and other multiple points of view, such as Jong's.

Participants' discussions transformed a teacher' personal teaching into case writing and made participants reflect on their teaching practices in-depth. Without such re-writing activity and group member support, teacher-written cases are unlikely to achieve clarity and power. Therefore, discussing the issues of learning and teaching of mathematics with those who teach the same content is a beneficial vehicle for constructing cases of teaching.

Conclusions

Cases which were conducted collaboratively by a school-based team consisting of the researcher and classroom teachers brought special benefits to the participants who wrote them. Constucting cases was an effective teaching method for teacher professional development because it provided the participants' with the opportunities to share insights and opinions in order to advance their knowledge and understanding. The case-development process prompted the participants to reflect on their practice and to become more analytical in their teaching. By constructing cases, teachers identified dilemmas of teaching and understood the purpose of the task, the presentation of the task, and the task as carried out by the students. Compared to an individual's case writing based on personal experience, sharing multiple perspectives and comments in a school-based professional development project was more likely to enrich cases as exemplars.

Teacher development in this study included the process of writing and rewriting cases which were based on real teaching. The use of cases, including both developing participant's conceptions about teaching cases and the processes of writing the case, enhanced teachers' understanding and prompted heir reflection. The question of how effective the cases developed in this study may be with teachers who were not involved in the processes of constructing the cases is worth further investigation, and will be a focus in the next stage of the study.

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Indispensable Mathematical Knowledge – IMK and Differential Mathematical Knowledge – DMK: Two Sides of the Equity Coin

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There are many pedagogical approaches that genuinely strive to foster equity in mathematics education. In this paper we first analyze why, although these approaches do acknowledge diversity, equity is not really accomplished. We then describe our research-based TAP (Together-And-Apart) approach that has been implemented in two projects in very different contexts, ISTAP in Israel and MALATI in South Africa. We describe how TAP achieves equity by both acknowledging diversity and ignoring diversity thus disarming school-mathematics of its traditional role as the gatekeeper of students' future. Finally we focus on a research site and one teacher's struggles and achievements in his attempt to accomplish TAP's goals.

Most mathematics educators accept, and even justify, using mathematics as a gatekeeper of further education. Nevertheless, they are aware that the current practice of mathematics education "contributes to the regeneration of an inequitable society through undemocratic and exclusive pedagogical practices..." (MEAS1 Proceedings, page 3). They suggest, however, that this inequity in mathematics education be dealt with via the employment of miscellaneous methods of instruction and class organization.

The most widespread approach for dealing with inequity and students' diversity is ability-grouping, either by setting up 'same-ability' groups within the same class or by placing students with different abilities in separate classes. Research indicates that teachers view ability-grouping as the best way of improving the scholastic achievements of all students and as the only 'fair' way for dealing with students of different ability-levels (e.g. Oakes, 1985). *"The question of how early some form of instructional grouping of students should occur...My response would be...as soon as the teaching and learning of mathematics occurs..."* (Dialogues, November 1998). Recent research, however, has clearly shown that the tracking systems contribute to the regeneration of an inequitable society. Studies of this sort have concluded that the placement of students in ability groups, in and of itself, increases the gap between students beyond what would be expected on the basis of the initial differences between them (e.g. Linchevski & Kutscher, 1998; Slavin, 1990).

Other attempts have tried to support equity by designing learning environments that permit and encourage different levels of mathematical knowledge and sophistication within the same community of learners. They suggest that the way to cope with within-class inequity is by developing learning environments that are sufficiently flexible to allow all students to show what they know and can do (MSEB, 1993). In our view, however, the latter attempts and its practices deal with only one aspect of the equity principle. What they actually do is legitimize different levels of mathematics without taking into account the gatekeeper-effect of mathematics.

What actually happens in most of the above-described systems is that diverse levels of mathematics are legitimized in the early stages of students' mathematics education. However, at a certain point in time (which may be different in different systems) certain specific mathematical knowledge is required in order for the student to be accepted into a prestigious learning trajectory, for instance allowing the student to study in a mathematics class leading to an 'accepted' high-school diploma. This filtering process occurs more than once during the students' learning career. The students who have learned in the lower tracks in the tracking system or in alternative 'tracks' in the heterogeneous system find themselves unprepared for these critical moments.

In this paper we describe our research-based TAP (Together-And-Apart) approach that was developed in Israel and has been implemented in two projects in very different contexts, ISTAP in Israel and MALATI in South Africa. We first describe the major assumptions and guidelines of TAP. We describe how this approach genuinely supports equity not only through appropriate learning environments, but also by providing learning interventions that prepare students for their mathematical crossroads. We then focus on one teacher's struggles and achievements in his attempt to accomplish TAP's goals.

TAP's major assumption

Tracking systems violate equity. We believe that equity in school mathematics can be achieved only when all learners are members of a fruitful, diverse mathematical community. We realize, however, that a rich learning environment in and of itself cannot guarantee each member genuine school-mathematics. We also know that such a community can be mathematically productive and endure to the satisfaction of all its members only if on the one hand its members have sufficient shared mathematical knowledge to make meaningful interaction possible, and on the other hand there is enough space for all members to express their mathematical diversity and to experience success. For this to be fulfilled the learning environment should guarantee each student's IMK and DMK:

- Certain essential mathematical knowledge (henceforth called 'Indispensable Mathematical Knowledge' or IMK) should be owned by all students notwithstanding the acceptance of diversity in other parts of their mathematical knowledge. Indispensable Mathematical Knowledge is that part of genuine school-mathematics that enables the heterogeneous mathematical community fruitful interaction to the satisfaction of all its members, culminating in open doors to higher education. It also enables the students to cope with future activities in mathematics and with society requirements. Equipping each student with IMK supports equity by enabling all students to be full partners in the heterogeneous mathematical community. In our view, it is the teachers' duty to identify IMK as well as to identify students whose IMK is insufficient, and to take responsibility for providing these students with repeated opportunities for acquiring it.

It is clear that the mathematics curriculum and thus IMK might vary among different educational systems. It is also clear that the choice of curriculum is one of the mechanisms certain systems use as a filtering device. However, for the purpose of this paper we assume that the curriculum-derived IMK is given and justifiable.

- Students are entitled and should be encouraged to fulfill their differential mathematical needs, abilities and preferences (henceforth called 'Differential Mathematical Knowledge' or DMK). This should lead to the construction of a learning environment that accommodates differences in the ways learners think about, construct and display mathematical knowledge and understanding. It should lead to the design of a teaching model that responds to students' diversity.

In our view the above-introduced requirements can be realized only if the learning environment is designed to concurrently recognize diversity in students' 'entry' points but also, at certain carefully defined points in the learning process, sometimes to 'ignore' diversity: In these cases TAP "does not accept" diversity in students' exit points. Ignoring diversity means that IMK should be owned by all students. Thus, ignoring diversity should lead to the design of a learning environment that guarantees students' acquisition of IMK (For more details see Linchevski & Kutscher, 1996 &1998.).

The research site: Stonehill High

As previously mentioned, TAP has been implemented in two different countries, ISTAP in Israel and MALATI in South Africa. A report and description of TAP's success in accomplishing equity in ISTAP as measured by students' mathematical achievements may be found in Linchevski & Kutscher (1998). In the current paper we report on the implementation of TAP by MALATI in Stonehill High School, South Africa.[1]

Stonehill High is one of seven schools participating in the MALATI Project in South Africa. This school is situated in a traditional black township and is, in many ways, typical of schools in disadvantaged areas in South Africa. The class-size at Stonehill High ranges from 40 to 50 students per class where students frequently have to share desks and seats. A considerable portion of teaching time at Stonehill is lost due to administrative reasons. For example, students' registration and time-tabling is finalized only at the beginning of the school year. At the beginning of the 1998 school year ten school days were used for the latter purposes. Teaching-time at Stonehill is disrupted on a regular basis mainly due to administrative issues and school events. More learning time is lost during the weeks that are devoted solely to examinations. All time lost is not compensated for.

Classroom practice at Stonehill prior to working with MALATI was typical of that in South Africa and elsewhere: Lessons were teacher-centred with whole class teaching the norm and dominated by low level questions and the mastery of procedural skills (Taylor and Vinjevold, 1999). Teaching was authoritarian with very little room for analysis or critique. Students at Stonehill were not accustomed to working in groups – they did not listen to one another and struggled to communicate orally or in writing. There was also little culture of doing homework at the school.

Assessment at this school was typical of the wider practice in South Africa. It was exam-driven, with "control tests" occurring at the end of a section or school term, and was used for reporting purposes (Niewoudt, 1998; Taylor and Vinjevold, 1999). The curriculum is divided into sections so that at the end of each quarter examinations were administered for each grade separately culminating in final, end-of-year examinations that assessed all the material learned throughout the school year. Promotion of students from grade to grade is not automatic a considerable number of students in each grade at Stonehill are failed each year. These students, nicknamed "repeats", are required to repeat the entire year of schooling. The pass rate for mathematics in the final year (grade 12) at this school is very low.

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[1] The name of the school has been changed.

Prior to the MALATI intervention, no IMK identification or consolidation[2] took place. No DMK diversification[3] between students was carried out. There was no attempt to follow up students' difficulties nor to take responsibility for bridging essential gaps. No analysis about what was and was not crucial for understanding subsequent topics was done. The teachers seldom used other forms of assessment such as projects or oral assessment.

Compounding these difficulties is the political background from which the Stonehill teachers stem. Complicated problems that are the product of the recent emergence of equality due to political changes in South Africa must all effect the teachers' grapple with MALATI's concept of education and, especially, equity. As members of a society previously discriminated in South Africa, these teachers had been part of the struggle for equality and democracy. Despite this, in their role of mathematics teachers they unwittingly continued to practice in their schools all the elements of undemocratic pedagogical practices, analyzed and discussed in this paper. And if in the past few years there has been more awareness (fostered through constant exposure to the state's new philosophy of a learner-centered, outcomes-based curriculum) that the school's pedagogical practice regenerates inequity within the school (and thus, eventually, jeopardizes its students' future) through its undemocratic methods, the teachers usually blamed outside forces for these problems, and expected external interventions to assist them in solving their problems.

The state's attempt to redress past inequalities in the distribution of human and physical resources has resulted in uncertainty that has led to an exodus of teachers from the profession and low-morale amongst those that remain. Unlike many South African schools, Stonehill has a mathematics department that has changed little during recent staffing changes. The six mathematics teachers at Stonehill have taught at the school for at least 7 years. Prior to MALATI, mathematics departmental meetings mainly dealt with administrative issues.

The decision to form a partnership with MALATI was taken by the whole school mathematics staff. This decision was facilitated by the fact that TAP is in line with the state's new philosophy of teaching and learning. The mathematics teachers seemed to be open to innovation and change, and participated enthusiastically in the MALATI project, forming a cohesive unit from its introduction.

MALATI supported the Stonehill teachers by providing learning materials, by its counselors' frequent visits to their mathematics classes and by weekly workshops where the teachers discussed appropriate strategies for cooperative learning, assessment and class organization in their heterogeneous mathematics classes according to TAP's principles.

In the context of the above factors and difficulties we now present a case study of Mr L and his attempts to implement TAP over a two-and-a half- year period. We will report in more detail on his first two years until his major break-through.

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[2] Process of guaranteeing IMK to students in need, after assessment indicated that their IMK was not yet acquired.

[3] In this article diversifying means organizing the class in homogeneous groups in order to cater for the differential needs of the students

Mr L:
Mr L was a competent teacher with full command of his class. He felt most comfortable in his role as "center-star" in his teacher-centered classroom. But, as we will soon see, this quality interfered with his success in adopting and implementing more learner-centered environments. At this stage of MALATI, Mr L's perception of teaching mathematics meant demonstrating the solution process of an exercise and thereafter practicing it for a predetermined period of time. At the time he believed that: *"The answer is more important than the process"; "when a pupil can use a mathematical procedure correctly he understands it"; "if students methods are inefficient they* (these methods) *should not be encouraged"; "mathematics tasks can be solved only in one way".* His main source for exercises was the textbook, usually inspired by the type of exercises these students would solve in their matriculation examinations - given they would reach this stage. This practice was dominant regardless of the students' grades, knowledge or success. The fact that many students failed, and that most students did not reach 'matric', did not trigger any process of reflection in Mr L other than devoting more preparation periods before end-of-term and end-of-year exams. Mr L did not believe in group-work. His guiding philosophy was: *"pupils cannot solve mathematical problems effectively unless they have been shown how to do them"*. He believed that assessments should take place at prearranged times that were decided on at the beginning of the year; it was not necessary to consider *"when pupils or teachers feel that the pupil is prepared"*.

1997: First half year, July[4] - November: In the first half year, interaction between MALATI and Stonehill was limited to discussions in workshops.

1998: 10th January, Second year: Mr L's attempts at TAP started by organizing the class for group work. The children were grouped randomly because *"I feel I don't know them well enough yet."* Despite the groups, the learning was whole-class and teacher-centered.

Six weeks later Mr L was trying to initiate improvements in the school. He had submitted a number of proposals for ways of stabilizing the timetable but to no avail In the TAP spirit, after the first evaluation he reorganized his class into heterogeneous groups based on the test results Although he went through the motions of fine-tuning the group composition as if readying pupils for 'real' group-work, he was unable to relinquish his role as 'center-star': The IMK consolidation was done in a whole-class setting.

9th March: Mr L reported that when marking the exam papers he noticed names of pupils he didn't recognize from class (even though they obviously attended his classes). And two of the latter students had outperformed his others pupils! He strode into this class, sought out these 'unknown' students, and then proceeded to lecture his class on their lack of motivation and hard work. The pupils had been introduced to the topic of "Probabilities with Dice" and since the MALATI tasks were inquiry-oriented and not procedure-and-drill, the students were not taking them seriously. Mr L commented to the pupils that some of them were complaining that the (mathematics) work they were doing was just a game. He assured them that this was still mathematics, with a new approach. Mr L's views of mathematics-learning were apparently starting to change.

Three months into the school year: Important TAP changes could be observed in Mr L's class. He was reviewing their control test. After giving them a pep talk on the importance of mathematics for future employment, he divided them into two groups[5] according to the information derived from this test's profile. Those students who needed IMK consolidation were divided into homogeneous pairs; the rest formed a small group at the back of the class. The students moved willingly and quickly. He found teaching in this learning environment *"a scary process"*. While he was doing IMK consolidation, the other group continued with other activities and *"continued and worked very quickly, and then I was split again and I still have difficulty handling the different levels that the learners are at. I find that some people can finish their activity quite quickly, and then they have a negative input, or they become playful..."* His practice had undergone a major change: the tasks were not procedure-and-drill. However: *" They* (the children) *don't like it because it's 'easy' and 'different'. The kids are so used to struggling with maths that they don't know how to handle it"*. As was apparent on the 9th of March, we observe here too that Mr L had to contend not only with his own difficulties in the process of his changing views of mathematics learning, but also his students' difficulties, all having come from a completely authoritarian culture of learning in general, and steeped in a mathematics culture of procedure-and-drill in particular.

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[4] The school year starts in January.

Two weeks later: The children were still sitting in their homogeneous pairs designed previously for IMK consolidation, although they had started a new topic – geometry. He handed out an activity and gave them five minutes to tackle it. Most of the children struggled (not having stronger students to confer with in their pairs) with the activity. Just when they got going, they were told time was up and Mr L initiated a whole-class discussion.

30th August: Slowly he became aware of his shortcomings and tried to be more conscious of the time devoted to group-work. He explicitly encouraged the students to work more independently in their groups. He was beginning to trust the students' abilities of learning: *"Many times I will leave them, but I will leave them with that doubt that I am not happy, so they will see where the problem is, if any, but I don't like guiding them in that direction."*

10th October: Mr L appeared quite comfortable with group-work. After assessment he again diversified the class for IMK consolidation. He prepared extra activities for those who performed well while he himself interacted with the others. Pupils got down to work quickly and continued so most of the lesson. When asked how he felt, he said: *"I think it helped the 'front'* (consolidation) *pupils. But I think I will have to assess them again to be more sure."* He was becoming more convinced of the benefits of group-work and of diversifying: *"In the smaller groups I find that I can be more attentive to them whereas the others who I feel don't need that much attention can go ahead."* He was gaining confidence in the ability of children to work on their own.

3rd November: Mr L was very frustrated by his lack of success in having the students do their homework and he constantly expressed his disappointment. He maintained group-work and some diversification but started to express dissatisfaction: *"I might be neglecting the stronger pupils and I need to work on this"*. His beliefs and attitudes seemed to have undergone major changes. He was aware that he was battling on three fronts: 1) the children's views of what mathematics is; 2) school and department regulations; 3) his old practices and beliefs.

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[5] This means that there are two different, concurrent learning-plans so that every small group is involved only in on of the learning-plans.

1999: First week, January, third year: The school was still not organized for scheduled learning due to administrative reasons. Mr L arrived to attend his second lesson but an unplanned administrative session that morning took more than an hour so all classes were shortened. He said that he had proposed to the principal that periods should be no less than 50 minutes but the principal had some objections. Mr L felt he could handle the class in the TAP spirit and would need assistance only after the first assessment.
11th February: Once again learning time was wasted on administrative purposes thus the lessons were very short. The students were sitting in rows while Mr L was conducting a traditional teacher-centered class. As the lesson progressed he gradually encouraged the "quickers" to pair off or form groups which they did quite readily. He was concerned that *"the learners who work very quickly will get frustrated when working with slower learners"*. The counselor realized that Mr L was implementing mainly homogeneous groups. He suggested that the learners work in heterogeneous groups, at least for core activities, so that all could benefit from the interaction. Mr L seemed determined to work on it (and on himself).
9th March: Following the end-of-term exams Mr L maintained homogeneous groups even after IMK consolidation for a topic was completed. He indicated that the 'repeats' had benefited from IMK consolidation. The counselor urged him to implement heterogeneous groups especially for the core material.
One month later: Mr L indicated that the 'repeats' had given up on mathematics. So he decided to try to integrate some of them with the rest of the class[6].
22nd April: It seems that Mr L had made himself the commitment of adopting heterogeneous groups as his dominant class-practice. The class was organized in mixed-ability groups and Mr L moved from group to group struggling to get the learners to compare their answers. During class discussion the different groups had to report back. He had adopted the practice of sitting with a group for an extended period. This personal interaction with the students may explain why this year he knew not only the strategies used by the different groups but also by the different students, whereas at approximately the same time the previous year he barely knew his students' names.

From this point on Mr L's practice was focussed on fostering cooperative group-work in mixed-ability groups within a mathematical culture of inquiry and discourse. It was clear that Mr L was able to and indeed did successfully implement most of TAP's principles. He no longer saw himself as 'center-star': *"They can learn from one another, that is – is what I have learnt... they can also learn from me."* There is no better way to describe the change than using Mr L's own words: L: *"Perhaps I am scared because it worked..."*; (Mr L and the counselor both laugh) C: *"Why are you scared?"* L: *"Because they don't need me"*. However, it was also evident that he was still oscillating between his old beliefs and his new experiences: He was not yet a full partner of TAP. He consistently needed proofs that TAP strategies were truly beneficial. At times he provided different groups with different activities but once the students were involved in these activities he began to have second thoughts, focussing on the negative aspects on what each group missed by not doing the others' activities instead of realizing what they had gained having had their specific needs addressed. At other times he would see a positive aspect and then seem to draw back as if to reinforce his original beliefs. He would declare that weak learners could benefit from working in mixed-ability groups: *"Uhm, what I'm finding is that in many groups certain people adopt those people that are not performing well..."* and in the same breath he could say that he was not sure that the weak students benefited from learning in mixed-ability classes. It was clear to him that mixed-ability group-work benefited the "strong" learners: *"I find that when they* (the "strong") *communicate in the* (mixed-ability) *group they also learn some other skill – of speaking mathematics, which is of great help for them"* - echoing Vygotsky (1986). And again a need to retract: *"I need proof that the strong learners would benefit from working in mixed-ability groups"*. If previously Mr L was concerned that he *"might be neglecting the strong pupils"* when they learned independently in the homogeneous groups, he now believed that *"within the group there is over enough intelligence to actually run through the activities."* But he still had a problem of *"a difficulty of the letting of one group go ahead."* This last difficulty was not only one of class-management and logistics that he was still experiencing. These expressions of contradictory beliefs were characteristic and representative of the way he expressed and exposed his inner conflicts with TAP's principles and practice.

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[6] The traditional practice at Stonehill was to sit the repeats at the back of the class. They were usually physically much bigger than the 'regular' students in their class.

When summing up, we can see that even under the objective difficulties – school culture, facilities, students' learning culture and the like – Mr L's practice underwent a remarkable change in terms of TAP. But from the many discussions and interviews, it was apparent that his beliefs did not undergo the same change. Why would a teacher with so much evidence, even hard data (*"looking at the results of last year versus the results that they obtained thus far... out of a class of 48 only five people have not improved on their mark of last year"*) and with a successful record of implementation, still cling to his old beliefs? Along with all the commonly recognized factors that affect beliefs, such as the change-agent's role, beliefs lagging behind practice, personality etc, one cannot ignore the social-context factor in which Mr L's change occurred. Most of the aspects in Mr L's old practice, such as teacher-centered lessons, end-of-term 'control' tests, 'failing' students becoming 'repeats' etc, had been shared also by previously privileged S.A. – and it seemed to work for them! Thus from this standpoint it might be reasonable to believe that with inequalities redressed and improved resources, most of the problems that the school experienced would disappear. Taking this perspective, it might be very difficult to be convinced that it was the old practice that posed the problem. Only honest reflection on the old practice will allow change where beliefs and practice go hand in hand.

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PROBLEM POSING AS A TOOL FOR LEARNING, PLANNING AND ASSESSMENT IN THE PRIMARY SCHOOL

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Abstract: *This paper describes and discusses the way in which a class of Grade 3 children (aged 8-9 years) constructed and designed mathematics problems for other students to solve. Individuals were required to pose appropriate tasks for students at grade levels below and above that of a Grade 3 student. The way in which the children engaged in problem solving prior to, and after, formulating or posing a problem was explored. Insights into the children's mathematical abilities were identified through the problem-posing activities. It is argued that such activities are a useful tool for planning and assessing in the classroom context.*

Introduction

Problem posing is an important companion to problem solving and lies at the heart of mathematical activity (Kilpatrick, 1987). The National Council of Teachers of Mathematics (NCTM, 1989; 1991) have called for an increased emphasis on problem-posing activities in the mathematics classroom. Problem posing has been used to refer both to the generation of new problems and to the reformation of given problems (Silver, 1994). In the first instance, "the goal is not the solution of a given problem but the creation of a new problem from a situation or experience" (Silver, Mammona-Downs, Leung & Kenney, 1996, p. 294). Until recently, few studies have examined the mathematical processes employed by the problem poser when constructing a problem. As Silver (1993),commented:

> despite the interest, however, there is no coherent, comprehensive account of problem posing as part of the mathematics curriculum and instruction, nor has there been systematic research of mathematical problem posing. (p. 66)

More recently, some educators have recognised the importance of promoting problem-posing opportunities in the curriculum (English & Halford, 1995; Leung, 1996; Lowrie, 1998; Silver, 1995). These studies suggested that when proposing a problem to solve, the originator of the problem will usually investigate the type of processes required to solve the problem. Importantly, the solution to the problem is likely to also be considered.

Silver (1995) identified four types of problem-posing experiences that provide opportunities for children to engage in mathematical activity. He argued that problem posing could occur *prior* to problem solving when problems were being generated from a particular situation, *during* problem solving when the individual intentionally changes the problem's goals or conditions, or *after* solving a problem when experiences from the problem-solving context are modified or applied to new situations. The way in which children engage in problem solving prior to, and after, constructing or posing a problem was investigated in the present study.

Stoyanova (1998) identified a number of categories that could be used by teachers and researchers to identify different problem-posing situations. These categories included: a) free; b) semi-structured; and c) structured problem-posing situations. One of the situations described in the free category included *problems written for a friend*. In such cases, a student creates a problem for a friend to solve. Some researchers (Ellerton, 1986; Mamona-Downs, 1993) have found that, for motivational purposes, it is helpful to have someone in mind when designing problems. In the present study, the friend would be someone in the class below or above that of the problem poser. As well as generating a problem for someone to solve, the Year 3 problem generators would also be asked to articulate why such a problem would be appropriate for the respective problem solver. The feedback obtained from this stage of the process fosters a reflective component of the problem-solving process (Lowrie, 1999; Silver *et al*, 1996).

Ellerton (1986) found that encouraging students to write problems for a friend was a useful way of understanding that person's mathematical ability. In such problem-solving situations the problem poser is forced to consider the individual for whom they are designing the problem. As Stoyanova (1998) commented:

> there is a strong acceptance among researchers and educators of the notion that students' ability in posing quality problems provides a useful indication of potential mathematical talent. (p. 172)

The very fact that a student must consider the mathematical ability of another person when engaged in free problem-solving situations requires reflection and careful planning. In order to complete the task successfully, the problem poser might not only focus on the underlying structures of the problem but also the extent to which the problem solver will be able to interpret the components of the problem. Such metacognitive thinking processes encourage mathematical power. Some educators (eg., Kulm, 1994; Leung, 1996) have argued that learners as assessors compliment and even enhance a teacher's understanding of students' mathematical ability. Such assertions are also explored in this paper.

It could also be argued that problem-posing situations allow children to have some control over the curriculum content and the type of learning activities presented in the classroom. Furthermore, the tasks or the activities children construct may provide insights into their beliefs or attitudes they have toward mathematics and the way in which mathematical knowledge is developed.

Methodology

24 Grade 3 students (ages 8-9) were asked to design mathematics problems that could be solved by students in Grade 2 and Grade 4. Although the students were not

asked to generate a problem for a particular individual (Ellerton, 1986; Lowrie, 1999) the intention was that the problem should be for someone mathematically less able (Grade 2 problem) or more able (Grade 4 problem) than the problem poser. Thus, the problem poser needed to consider the appropriate content and process that would be required for another individual to engage in the task.

The students did not have much previous experience in generating problems for others to solve, however they were accustomed to communicating their thoughts about problems they solved. The classroom teacher regularly used direct questioning techniques and brainstorming sessions to encourage the students to process information metacognitively, and as a result individuals had considered relationships between content and processes when completing tasks on a regular basis. As well as generating a problem, each student was also asked to describe why they felt their problem would be suitable for either a Grade 2 or Grade 4 student to solve. The following commentary provides details of these first attempts by Grade 3 students at generating problems for students of both lower and higher mathematical abilities than themselves to solve.

Results and Discussion

The written accounts, which provided a justification for why the Grade 3 students felt their problems were appropriate for each of the other grades to solve were categorised according to the following five elements: a) magnitude of the numbers used; b) appropriate content for the grade level; c) learning to solve problems; d) interests and real-life experiences; and e) open-ended tasks. Each of these elements is addressed below and includes examples of both the problems generated and the explanations concerning their suitability by the Grade 3 students.

Magnitude of the numbers used

Responses within this category consisted of problems where the numbers used were increased or decreased in order to change the difficulty of the problem. In essence, this meant that students considered the content of the problem and matched this content in reference to their personal benchmark, that is, their own level of understanding. As such, the problems Grade 3 students intended for Grade 2 students included smaller numbers than they themselves found challenging, whereas the problems for Grade 4 generally included numbers the students were unable to use confidently in calculations themselves.

Claire, for example, generated the following problem for a Grade 2 student to solve:

> hansel and grettle ate 6 lollies each there where 20 lollies how many does the witch have left

Claire explained that this problem would be appropriate for Grade 2 students:

> *because the numbers aren't to big and its easy to work out*

This type of response was also observed in the problems individuals generated for Grade 4 students to solve. Students were not always able to provide a solution to these problems but were confident that it would therefore be an appropriate and challenging problem for Grade 4 students. Emily's problem, and her explanation of it, were coded in this category and appear below (students' problems and their explanations of why they are appropriate will continue to be reported in this way throughout this paper).

> There were 490,000 cherries in the tree and 48 birds came and picked 1 cherry each how many cherries are left? 410,000
>
> *because it is a little bit harder than the year two problem.*

Appropriate content for Year level

Although the structure and content of problems in this category were quite varied, each shared the common feature of relating to content that the Year 3 students saw as being appropriate for each of the grades. Thus, the students endeavoured to provide problems that would enhance learning.

Cathy's problem for Year 2 was coded in this category:

> if Sally weighed 50kg how much would Molly weigh if she was half of her weight?
>
> anwer:25kg
>
> *They should now about kg and they should now half of things.*

The problem posed for Year 4 students by Lillian was also of this type:

> If there were 11 hens and each laid 12 eggs. How many eggs would there be altogether?
>
> *I think this is a good problem for Year 4 because Year 4 should know their 11x tables.*

Learning to solve problems

The students who generated problems in this category showed an awareness of the need to develop and encourage problem-solving skills. It is important to note that children who generated problems of this type were highly proficient problem solvers themselves. It appeared to be the case that these children designed their tasks on foundations that required a specific strategy to be used, such as Harriet's problem for Year 2:

> If there was twenty-four diarys and I had to shair them out with three people. How many diarys would they get each.
>
> *Because I thought that they would like to draw a diagram to help them work it out. I made shore that it wasn't to hard or to easy.*

Other responses, such as the task designed by Cathy for a Year 4 student, also displayed the understanding that multi-step problems increase task complexity:

> if a cake costs $2, Lemington costs $1, fanda costs $2.98, mose cost $3.75. what would it cost when I get: 5 cakes, 10 lemingtons, 7 fandas and 14 moses?
>
> *Couse I don't know the antwer. Couse I made the prices high. In the question I made lots of the food together.*

Interests / real life situations

A small number of students designed tasks that considered the interests and possible real-life experiences of students of different grade levels. The students incorporated these factors in a variety of ways. Although some students merely referred to common interest areas (e.g. using teddy bears in problems for Year 2 students), others based their problems around playground experiences (e.g. common school routines) or areas of study being investigated by other classes.

Christina's problem for Year 2 was based around playground experiences:

> Q. if I had $3.50 and licksticks cost 20c how much chang will I get.?
>
> A. $3.30
>
> *because it is esey and it would be esey because licksticks do cost 20c at the cantent*

Claire's problem is based around a topic currently being investigated by Year 4:

> I have being to 4 different contenents in Australia how many more do I need to go to before I've been to all of them
>
> *because its hard and I think that's what there learning about*

Open-ended tasks

Students who constructed problems coded in this category were also highly proficient problem solvers. These individuals appeared to acknowledge the usefulness of open-ended tasks in catering for students with a diverse range of abilities. Importantly, only two students responded in this manner. Both of these problems were posed for Year 2 students. It appears that this was an easier process when generating problems below the students' own level of understanding than above.

Due to the complexity of posing open-ended tasks, it appears that students need to have a higher level of understanding about both problem solving and mathematical content than those they pose problems for if they are to be successful in designing tasks of this type. Hilary's problem for Year 2 provides us with an example:

> Kirsty can't think of a problem for Year 2. Can you help her?
>
> *Because they make it there own level of difficulty.*

Problem Posing as an Assessment Tool

All of the students in the class considered at least one of the five elements identified above when posing problems for others to solve. On some occasions, students integrated more than one of these elements into their problem design. Interestingly, two students employed the same technique when posing problems both above and below their own benchmark. This consistency provides the teacher with a rich source of data which can be used when designing appropriate learning tasks for the student (Lowrie, 1999).

Alexandra, for example, posed the following two problems for Years 2 and 4 students respectively:

> If Rosemery had 15 lollies and shared them with Sara, Samantha how many would they get each.
>
> *I think Year 2 should learn about problem solving and how to solve my problem. I would divide it into groups like 5 5 5 and see how many you would get. I had 56 lollies but i changed it to 15 because i thought it would be too hard because there would be some left over.*
>
> If Hannah had 104 lollies and she gave Rachel 34 and Anna double times that who has the most lollies Anna Rachel or Hannah.
>
> *I think this is a good problem for Yr 4 because it is a bit eazy for Yr 5 and a bit hard for Yr 3 and did have 87 lollies but i thought it was a bit too eazy so i changed it to 104.*

From these worksamples, a picture of Alexandra's knowledge about problem solving as well as her understandings of division concepts can be drawn. With respect to her understanding of division, it is evident that she understands the notion of 'equal shares'. She can explain that 56 is a difficult number to share among three people "because there would be some left over." Moreover, she appreciates that 15 shared among three is more appropriate for Year 2 students as it can be evenly shared into groups of a manageable size by "divid[ing] it into groups like 5 5 5 and see how many you would get."

Part of Alexandra's reasoning for posing her first problem is to encourage Year 2 students to "learn about problem solving." As she is comfortable with the content of the problem, she is able to suggest an effective strategy for completing the task. However, when she has been required to develop a task at a higher level than her personal benchmark, this same knowledge of problem-solving skills and strategies is not evident. This could be due to Alexandra's belief that those in a higher grade already know the things she knows about problem solving. Nevertheless, important information about her current skills and understandings can be established from this more difficult problem. She has attempted to incorporate her developing concepts of proportion and the links between multiplication and division. In addition, Alexandra has selected much larger numbers for the higher grade student to contend with. She stated that she "did have 87 lollies but [she] thought it was a bit too eazy so [she] changed it to 104." We maintain that this information indicates Alexandra's readiness to move from concrete division problems involving equal sharing with relatively small numbers to more sophisticated concepts involving proportion.

Through the two problems she has generated, Alexandra has provided us with a clear picture of content she has mastered as well as her own learning goals. Although we are not suggesting that she is able to complete such problems at present, her ability to pose such tasks may be an indicator of readiness in this area.

Conclusion

All of the Grade 3 students investigated in the study were able to generate problems for others to solve. Decisions concerning the appropriateness of content for a particular task were often related to number magnitude, operation complexity or the type of mathematics concepts thought to be taught at a grade level above or below that of the problem poser. Other problems were embeded in contexts that were related to perceived interests or real-life experiences problem solvers had encountered. Other decisions about task appropriateness focused on specific problem-solving strategies identified as being important at particular grade levels. The problem-posing activities allowed the classroom teacher to gain insights into the way students constructed mathematical understandings and served to be a useful assessment tool.

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LENGTH MEASUREMENT

CONVENTIONAL UNITS ARTICULATED WITH ARBITRARY ONES

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Abstract

This paper presents the importance of learning the processes of construction of a length measure, using non-conventional instruments and arbitrary units, for students of the 5th year of the first grade, who know how to measure using conventional instruments of the metric decimal system. It presents the relevance of the teaching of measures which is not limited to the use of conventional instruments and representations in usual scales (1:10 or 1:100). It shows that a teaching which articulates the use of conventional instruments and non-conventional ones promotes a greater flexibility in the choice of the adequate instrument, unit and procedure to solve a problem.

1. Introduction

Researchers from different theoretical backgrounds have developed studies about length measurement. In classroom situations, with 9-12-year-old students, Douady e Perrin-Glorian (1986) study the learning of measurement construction of the grading of rulers, starting up from non-conventional instruments and arbitrary units, until arriving at the conventional ones, to promote the understanding of fundamental conceptions and procedures characteristic of a length measurement (in the usual Euclidian metrics). They consider that this knowledge is a tool for new learning, such as representations in scale. These authors have not studied, in special, situations in which the students could have the choice between the use of measurement units present in graded rulers and an arbitrary one, nor the effects of such choices, on the solution of problems involving representations in scale.

Nunes, Light e Mason (1993) state that the use of conventional measurement instruments, such as graded rulers, in the solution of problems involving measurement of length, is possible and advisable since the first year of the first grade. In that laboratory study they analyze the performance of students in the first years of the first grade, in tasks involving the comparison of lengths of segments. Such tasks are performed through the exchange of messages among the students in three distinct situations: a) one, in which they receive arbitrary units of length measurement (strings); b) another one, in which they receive broken rulers, starting at four centimeters; c) and a third one, in which they receive conventional instruments, that is, common graded rulers. What we want to mention here is that their conclusions emphasize that there is a significant difference in the performance of the students, in situation (c), in which they use the common graded ruler when compared to situation (a), in which they use non-conventional instruments. They conclude that the students benefit from the use of conventional instruments, especially from the numbers present in the common rulers, the former working as thinking tools for the solution of the problems proposed in their study. These authors have not studied the performance of students involving representations in scales, in which the solution using the units present in graded rulers, by conventional procedures of measurement, would be more difficult than the solution obtained using arbitrary measurement units (strings) and non-conventional procedures.

2. Theoretical Framework

Despite the fact that Nunes, Light and Mason (1993) do not emphasize the promotion of geometric knowledge, they state that the understanding of the invariants of measurement such as the use of the measurement tool, both of which support important reasoning processes related to measurements, will have a direct effect on the performance of the children in tasks of comparison of lengths, through measurements. From the analyses of this study, we point out that: in situation (a), using non-conventional tools, the study shows that 43% of the students transpose the string and that there is a rigorous subdivision of the same in 13% of the trials. Other procedures, not considered as rigorous ones, are observed, with statements such as these: "my line is about twice the string" or "I think my line is about 3 cm". A significant number of trials requiring iteration or subdivision is solved based on insufficient information and in an incorrect way; in situation (b), using the broken ruler, the authors consider that the students respected the principles of measurement in 63% of the trails. Only 20% read the number on the ruler corresponding to the end of the segment. The remaining 17% take into account the break in the ruler, but they reduce the number read at the end of the segment by 3 or 2 cm; in situation (c), using the common graded ruler, all students give numerical responses, being that some of them match the beginning and the end of the segment with numbers of the ruler and the other ones count segments between the numbers of the ruler (units). There are some errors which resulted from starting measurements from number 1 over one end of the segment to be measured, instead of number 0. We understand that all these errors made by the students may derive from the lack of knowledge about the processes of construction of a length measure.

The authors state that the errors of positioning the common ruler may be overcome through a better instruction on how to place the ruler to measure correctly. On the other hand, the non rigorous procedures with arbitrary units may result from the need to reinvent iteration and subdivision to obtain quantification. We agree with them and therefore we thought it convenient to teach such procedures. The processes of construction of a length measure, according to Douady and Perrin-Glorian (1986) may provide a good understanding about the numbers present in a graded ruler. As side results they would obtain, for instance: a) the knowledge of valid geometrical procedures such as the juxtaposition or transposition of segments to measure, thus deriving the understanding of the presence of number 0 in segments or in graded rulers; b) the conception that measuring a segment v (in a measurement unit u) is knowing how many measurement segments u cover v, making a distinction between numbers that designate points (abscissas) and numbers related to the counting of units. In short, from our standpoint, the view by Nunes, Light and Mason (1993) can be articulated with the one by Douady and Perrin-Glorian (1986) and we see a contribution to teaching, derived from such articulation, because the students would have more resources for the solution of problems.

This way, *a hypothesis that we tested in our study is whether the knowledge of geometrical measurement procedures can provide the students who already know how to measure with conventional tools a greater flexibility for the solution of problems, mainly those which involve representations in scale.*

3. Methodology

In this case study, we followed-up group of 16 students of the 5th year of the first grade, from a school in São Paulo, Brazil. These students had been in that school since the 1st year. They had received instruction on:

a) The use of graded rulers with units of the decimal metric system (the most common ones in the Brazilian culture), for the solution of measurement problems, since the 1st year; b) calculations with decimals, since the 3rd year; c) calculation of distances, through representations in scales (maps), in Geography and mathematics classes, in the 5th year.

The study was developed in three phases:

In phase 1, we conducted clinical interviews with 8 pairs of students. Each pair received a test-problem, to be solved with paper and pencil, in a consensus. They also received some graded rulers (the common decimal metric system ones) and folding cutout paper strips (without numbers or demarcations). We let them know that any material could be used. We obtained data from the manipulation of the tools, the discussions to solve the problem and the answer sheets.

In phase 2, the group of students received, from their regular teacher, some classes with a sequence of situations based, partially, on the studies by Douady (1986) for the construction of a measure of lengths. We chose only some situations, aiming at improving their knowledge about the processes for the construction of a length measures.

In phase 3 we applied the same test of phase 1, under the same conditions.

4. Procedures

4.1. Description of the situations of phase 2

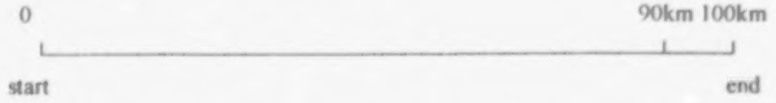
This phase was developed in 3 sessions of 100 minutes each, involving:

- The direct measurement, for comparison of two different lengths, but very close to each other, using any tools they wanted, including conventional and non-conventional ones; the answer to questions such as: "How many length units *u* cover a segment of a *v* length?", relating the answer to the measure of the segment in the unit considered; discussions about advantages and disadvantages of each procedure used by the students and about what is considered as a gross error of measurement, for each task;
- The grading of straight lines by transposing arbitrary units, the discussion about the reason why we use numbers 0 and 1 to designate the two first points of the grading; the demarcation of points on graded lines with arbitrary units and the calculation of distances; measures of some segments in different units, through grading; discussions about what is considered as a gross error of measurement, for each task;
- The construction of a 200-m straight racetrack and its division in 4 tracks of same length, for a relay race with a baton, in teams of 4 students each; the design of such tracks in scale, over a segment of 25 cm, which represented the total race course. The students could use folding 25-centimeter paper strips or graded rulers.

4.2.Conception of the test-problem of phases 1 and 3

The problem was conceived in such a way that its solution would be easier by using the paper strips, without demarcation. There were two strips, one of them 12 cm long, the same measure of the segment representing the 100 km of the route and the other one, which was 1.2 cm long, the same measure of the segment representing the 10 km of the route. By using graded rulers, the students would face the decimal numbers of the representation and their relation with the numbers of the real situation. The problem was like this:

A driver went for a 100-kilometer ride, driving at regular speed. Demarcate on the line below the points where he was at after he passed exactly 25 km, 55 km and 65km from the start.



5. Results

5.1 Phase 1 - The students spent between 40 and 50 minutes to solve the problem. Except for one pair, all the other students used the graded ruler and correctly measured the segments of the figure, as soon as they read the question. We transcribed the procedures and some reasonings used by the students.

3 pairs, A, B and C, formulated the relation *1cm equals 10 km* and **2 pairs, D and E,** formulated the relation *10 mm equal 10 km* and concluded that *25 km are represented by 25 mm*. Afterwards, one of the students, from **pair D**, formulated the relation *12 mm equal 10 km* and demarcated the whole segment subdiving it into 1-centimeter intervals, stating: *if we do it this way, it will be right*. Her companion agreed. **1 pair, F,** demarcated the whole segment at 5-milimeter intervals, stating that they so did *because 25 is a multiple of 5*. **The pairs A, B, C, D and F** demarcated the point corresponding to 25 km, 55 km and 65 km, respectively at 2.5 cm, 5.5 cm and 6.5 cm of the representation. **The pair E** demarcated the point corresponding to 25 km on 25 mm and calculated: *55mm + 2mm = 57mm*, demarcating the point corresponding to 65km at 57 mm and they demarcated the point corresponding to 55km at 56mm. **1 pair, G,** formulated the relation *1.5 cm equal 10 km* and, after, the relation *1cm point 2 equal 10km*. The student who used the relation *1.5 cm equal 10 km* made an error in both calculation and procedures for demarcation. When inquired, she added the measures of the 1.2 cm segment already demarcated, gave up and started to transpose the interval between 0 and 1.2 cm from the ruler, to demarcate a point corresponding to 25 km; the other one, who used the relation *12cm = 100km*, divided 12 cm by 4, obtaining 3 cm to demarcate the point corresponding to 25 km and, after, transposed a 3-centimeter segment to demarcate the point corresponding to 50km. She demarcated the point corresponding to 55 km at 1 cm from the point representing 50 km and the one corresponding to 65 km at 2 cm from the point representing 55 km. **1 pair, H,** formulated the relation *12 cm = 100 km* and *1.2 cm = 10 km* and, after transposed the segment between 0 and 1.2 cm from the ruler to demarcate the points corresponding to 25 km and 50 km. They transposed 0.6cm to demarcate the point corresponding to 55 km and 1.2 cm to the point corresponding to 65 km.

Except for pair **H**, all the others who used wrong relations between the representation and the real situation received interventions such as: "But, in the figure, 12 mm correspond to 10 km, don't they?" Only 1 pair revised their outcome. The remaining ones said that the figure should either have some kind of error or that the difference was too small or else that the way they had done so far was fair, which showed us that they were tired and not willing to reinvest in the solution. We told everybody that the use of the paper strips would make it easier to find the solution and suggested they use them, but all of them refused to do so.

5.2 Phase 3 - The students spent between 20 and 30 minutes to solve the problem. All of them used the graded ruler and measured the segments in the figure, as soon as they read the question. Afterwards, they folded the paper strip corresponding to the whole route, in half, and then in half again, for the demarcations, abandoning the graded ruler. The resolution of the eight pairs was like this: by folding the paper strip in 4, they demarcated 25 km. They either transposed this length or used the crease in the middle of the strip to demarcate the point corresponding to 50 km of the route. After they folded the strip corresponding to 10 km, in order to demarcate the point corresponding to 55 km. They said something like this: *To demarcate 55 we have two of 25 plus half of 10* or, *to demarcate 55 all we have to do is add half of 10*. Later, they unfolded the strip to demarcate the point corresponding to 65 km of the route. The reasoning transcribed in the following table show some of the choices of students from **pair D**.

D_1 – *Two centimeters equal ten kilometers; then, two millimeters equal one kilometer.*

D_2 – *Hold on, girl! Why calculate with the ruler? We have the strip to use!* (showing the paper strip corresponding to the whole route. She started to fold it in four).

D_1 – *If you always divide the strips in half, we don't have any lacks or surpluses and it always goes right. If you divide them in four straight away, we have a lot of surpluses.* (from this reasoning by D_1, D_2 changed the procedure).

D_2 – *To demarcate the twenty-five kilometers all we have to do is to fold the strip in two, and then in two again. After, we just have to use this little one here* (compared the measure of the paper strip and the segment corresponding to 10 km).

Except for pair **D**, all the others were inquired about the reason for not using the ruler. All of them answered in a similar way: *The calculations of the problem are bad because 1.2 cm correspond to 10 km and it is not a direct calculation.* When questioned about what direct would mean, they answered: *direct means using the length to see the distances. You don't have to calculate anything.*

6. Conclusions

In phase 1, none of the students solved the question by using a procedure that would eliminate the ruler and the measurements were all correct, without errors in the positioning of the ruler. Three students used the iteration already described by Nunes (1997), and **1 pair was correct in all demarcations**. We expected this because the students in our case study were in higher years and had already received instructions, in the standards suggested by Nunes, on how to take measures. So, this study enhances the results of that one, which points out errors in calculations or gross approximations when it required the use of fractions, showing that that occurs even with those students with more instruction on decimal numbers and how to take measures. Without intervention, **7 pairs made errors in the demarcations**, because they started from a gross numerical approximation, for their calculations, using relations which are common in representations in scale, such as 1cm corresponds to 10km, 10 mm to 100km (only one of these pairs used 1.5 cm corresponds to 100 km, and even so, just in one part of the problem).

The gross approximations of phase 1 could have been derived from lack of experience in the use of unusual scales, since most of the students expressed the correspondences (1:10 or 10:100) even after measuring the segments. The lack of notion about what could be considered as an unimportant error (in the approximations they made) in a representation in scale seemed to us to have also influenced on the behavior of the students, because when questioned they answered that the figure should have some kind of error or that the differences were unimportant. However, we consider that the lack of knowledge on how to use non-conventional measurement tools, favoring other procedures that would avoid numerical relations and calculations with decimals, was the main factor of the performance of the students in phase 1. That was corroborated by the performance in phase 3, considering the experiences they had in phase 2.

On the other hand, all demarcations in phase 3 were considered as good ones. We noticed more willingness and ease to solve the problem, from all of the pairs. There was a significant reduction in the time spent for the solution (from 20 to 30 min), since the students avoided relations between the measurement units of the real situation and of the representation, as well as calculations with decimals. The pairs showed more confidence in the solution of the problem, by choosing tools other than the ruler. That is explained by the several situations in which they had to use various measurement tools or procedures in phase 2. Such choice does not represent loss of confidence in the use of the graded ruler, because most of the students used it in the first part of the problem.

The choice of a measurement unit which is adequate to solve a problem is a relevant competence, from our standpoint. In this study we saw that this competence was also a reflection of the processes of construction of a measure, mainly because we worked with arbitrary units in measurement situations and also in situations of representation in scale, thus avoiding the use of the usual scales and the numerical relations between the real situation and the representation. We emphasize that the situation of representation in scale (in phase 2) involved working with just one measurement unit whereas the situation of the test-problem (in phases 1 and 3) involved the choice of two different units, for different demarcations, and these units kept the relation between the representation and the real situation. The students knew how to choose the most convenient unit for each demarcation.

The answers of the students, besides the observations, demonstrated an increase *of flexibility in the choice of tools which provided more economical procedures, for the solution of the problem, comparing phases 1 and 3.*

We believe that such flexibility derives from the articulation between the use of conventional and non-conventional tools, in problems involving unusual scales.

Finally we would like to highlight the importance of the students not being mere users of the tools which are available for them, but also knowing their production way. On the other hand, we emphasize the importance of the school appreciating the cultural background of their students, from the first year of the first grade.

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DYNAMIC TRANSFORMATIONS OF SOLIDS IN THE MATHEMATICS CLASSROOM

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This study explores how a number of teaching materials characterized by the dynamic transformations of solids operate in a mathematics classroom when geometrical solids and their properties are considered. The analysis focuses on the tasks and how they developed, on the development of children's geometrical thinking and on the teacher's and researchers' interventions. It appears that the classroom culture frames the way that these materials are implemented.

A number of research studies have investigated the role of dynamic environments in developing children's understanding of geometrical concepts and the changes that such environments bring to the kind of mathematical processes the children experience and construct (Laborde, 1993). Most of these environments are computer generated and aim to promote children's geometrical thinking on concepts in two dimensional geometry. Very few examples in the literature study the problem of children's thinking on three dimensional objects through the use of dynamic environments (Gutierrez, 1996; Markopoulos and Potari, 1999). Moreover, the research in this area is based on data coming from children working individually or in small groups in experimental settings. The problem of how the dynamic environments operate in the classroom environment still remains open.

The rationale of the study

In this paper, we extend our work with two pairs of children in which we had explored how these children built relationships between geometrical solids and their properties and between the solids themselves while working with dynamic three dimensional models (Markopoulos and Potari, 1999). In particular, we explore how a number of tasks based on the dynamic transformations of geometrical solids were implemented in a mathematics classroom supporting a "construction-oriented approach" . In this teaching approach the primary focus of the teacher is on the individual process of constructing or ascribing meaning, or producing presentations (Bauersfeld,1992). Focusing on the dialogic relationship between the dynamic models and the classroom culture means that the materials themselves are not "self-speaking" (Bauersfeld, 1995). We need to study the contributions of all the participants, - teachers, researchers and children, - as they interact in the classroom with the materials and the geometry curriculum. For example, the tasks and how these are adapted to the mathematics curriculum, the teacher's actions through his questioning and his interaction with the children, and the way children communicate are the elements that constitute this environment and probably promote children's geometrical thinking. Nevertheless the individual pupil remains our main focus. As Confrey (1995) suggests, we attempt to integrate both social and individual perspectives in our exploration of the mathematics classroom.

As it also appears in our previous study (Markopoulos and Potari, 1999) by dynamic transformation of a geometrical solid we consider a process where the solid changes its form through the variation of some of its elements and the conservation of others. This seems to be related to the "concept of invariance" which promotes intuitive reasoning (Otte, 1997, p.48

The process of the study

We distinguish three main phases in the research process. The first concerns observations of regular mathematics teaching in six primary school classrooms, three of the 4th grade and three of the 6th grade in three different schools in Patras, Greece. Participation in the study was on a voluntary basis. Each class was visited about 4-5 times over a period of two months. The aim of these informal observations was to get to know the classroom environment and familiarize the teachers with the materials and the rationale of the study and to build a rapport between the researchers, the teachers and the children. The research methodology in this phase followed the ethnographic research tradition (Eisenhart, 1988). In the second phase the materials were used by the teachers in cooperation with the researchers and took up 4-5 teaching periods for each class. This part of the research is a classroom teaching experiment along the same lines as described by Cobb, Yackel and Wood (1992). The plan was for the children to work on the tasks instead of following the official mathematics curriculum concerning geometrical solids and their properties. In the fourth grade the children had not been taught these concepts before. In the sixth grade they had already been introduced very briefly (for 2 teaching periods) to the recognition of the properties of the cube and of the orthogonal parallelepiped and of their construction, by using the nets. Finally, in the last phase the children of the 6th grade were interviewed one by one for an hour on tasks referring to an imaginary dynamic transformation of a cube and of an orthogonal parallelepiped.

The data from the first phase consists of field notes, informal discussions with the teachers, some audio recorded, and examples from children's written work on a given task concerning the solids. The data from the second and third phase includes video recordings which have been transcribed.

The data we analyze below comes from a sixth grade class where the teacher himself implements the tasks. One of the researchers intervenes either in cases where the teacher is in conflict or in cases where he wants to explore further and clarify children's contributions. The teacher, in his regular teaching used to adopt a rather traditional approach which was curriculum driven emphasizing children's right responses. He wanted to participate in the project as he thought that the materials would provide a good experience for the children. The analysis of our data is a bottom up process where issues emerged and were tested throughout the subsequent episodes in a similar manner to the one described by Cobb and Whitenack (1996).

The materials

The materials of the study are physical models, computer representations and written imaginary tasks. The physical models are: a number of three dimensional geometrical solids with fixed properties; a cube, the length of whose sides and angles could vary up to a certain size; an orthogonal parallelepiped where the length of its edges remain the same while its height varies from an initial length to zero when the solid "becomes" a plane; a transparent plastic cylindrical solid filled two thirds with salt. The computer representations were created on the Cabri Geometre II. These represent geometrical solids like cubes, pyramids, prisms and cones which can be varied without mechanical constraints on their properties. Examples of the materials are given in fig.1. The imaginary tasks referred first to an imaginary transformation of a cubic reservoir into a different one that holds a greater quantity of water and secondly to the identification of the transformation of an orthogonal parallelepiped to a cube.

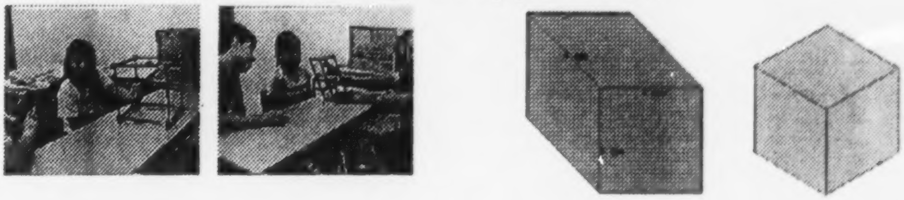

Figure 1 Examples of the materials

Results

From the analysis of the four episodes we present below three different aspects of the classroom environment: the tasks and how they developed, the development of children's geometrical thinking and the teacher's and researchers' interventions. Moreover, we try to see how these elements are interrelated. Through the episodes, we focus on a specific geometrical area which is the relationship between different geometrical solids. In the first three episodes physical materials were used while in the last the children worked on the computer representations of the geometrical solids.

Episode 1

In this episode the tasks used were based on static three dimensional models. The focus was on the identification and the recognition of specific geometrical solids and their properties encouraging the development and the adoption of a common geometrical language. The teacher's questioning was based on what the children could "see" in the solids and on their school knowledge. In some cases they developed a process of counting specific properties of the solids. Questions like "how many vertices does it have", "show me the edges, which are the edges?"," what do we call it in Geometry?" were posed. Towards the end of the episode the researcher takes the leading role. The tasks became more explorative when he asked the children to compare a cube and an orthogonal parallelepiped and find similarities and differences between the two solids. Then the children, conceiving them as different solids, talked about differences in their specific properties: "The cube has equal edges while the orthogonal parallelepiped does not ", "The six faces of the cube are all equal while in the orthogonal only the opposite ones are". The following response from a child in this class possibly shows a rather inclusive relationship between the two solids " the twelve edges of the cube are equal while the twelve edges of the orthogonal may not be equal". From the analysis of the second episode through the use of dynamic transformation this inclusive relationship becomes more explicit for the children.

Episode 2

In the second episode the material that was used was the cube with the "moveable" edges and angles. So, the focus of the tasks is on the identification of variation or invariance that the transformation caused in the solid or the transformation itself. In this framework, the different characteristics of the solids are explored. In table I we summarize a categorization of the tasks that were used in this episode focusing on the relationship between the initial solid, the generated one (the new solid) and the process of generation (the transformation) by giving examples of such tasks. The order in which we present the categories shows how the tasks developed through the episode. This order is repeated when the kind of transformation changes but it cannot be considered as linear. For example, the last three tasks are followed by a comparison between the two solids. The complexity of the tasks increases as we move to the last three categories as in order to complete the tasks the children have to anticipate the result of their actions. This demands not only the construction or the recall of the mental image of the initial solid but also its mental transformation, a process that precedes the actual physical action of the transformation. This could be more complex if the generated solid is also specified, as the construction or the recall of its mental image is also needed.

Categories of the tasks	Examples
Transforming physically the initial solid (a cube) into a different one through an unspecified variation of its properties	Children transformed a cube into an oblique parallelepiped by changing its angles.
Comparisons between the two solids highlighting the direct variations	Children transformed a cube into an orthogonal parallelepiped and compared them in terms of the equality of the edges
Comparisons between the two solids highlighting the indirect changes	Children transformed an orthogonal parallelepiped into a different orthogonal and tried to find out which one has the greater volume.
Transforming physically the initial solid into a different one through a specific variation of its elements	The transformation of a cube through the equivalent lengthening of its horizontal edges.
Specifying physically the variation needed to produce a particular solid	The equivalent lengthening of its edges in order to transform a cube into a bigger one
Specifying mentally the variation needed to produce a particular solid	The imagination of the way that an orthogonal parallelepiped could be transformed into a cube

Table I. Categories and examples of tasks used in the second episode

In this episode the questions asked by the teacher and the researcher in the first type of task focus on the recognition of the properties. Later when the tasks encourage the children to become conscious of their actions, the questions focus on those actions. Examples of such questions were: "What has become big?", "What has remained constant?". In some parts of the episode the teacher's tendency to "teach" the children the normal content of the mathematics curriculum concerning the solids appears. So, he asks questions about surface area and volume that are included in the normal curriculum. In the following extract the children have already made an orthogonal parallelepiped from a cube.

> *Teacher: So, from a cube we moved to an orthogonal parallelepiped*
> *Panayiotis: And now if we make it bigger but the same, we will have a cube.*
> *Teacher: Ah! So. If we make them all bigger but the same what will we have?*
> *R: Do you mean that we also have to make those on the top bigger?*
> *T: The same, yes*
> *Panayiotis: Now it is an orthogonal but if we make them higher , we will make a cube.*

It would appear that Panayiotis has reflected on his physical experience with the transformation and generated a cube from an orthogonal parallelepiped. The teacher and the researcher allow him to express his thinking and encourage him to share the idea with the other children in the class.

Episode 3

In this episode, the cube and the orthogonal parallelepiped were used. The tasks followed the same categories met in the second episode but here the questions require a reflection on the transformation. The questions change from "what are the changes?" to " how did you make all these changes?". However, in some interactions between the teacher and the children, the teacher does not built on children's experience. For example, the process of generating new solids is neglected and the emphasis is given on the "recognition" based on what the children "see" in the solids. Later on, the children are asked again to reflect on their actions with the materials and explore rather advanced geometrical concepts like the concept of volume. The kind of material in the orthogonal parallelepiped allows a gradual transformation of the solid to a series of different solids. So, the initial orthogonal parallelepiped becomes oblique and finally flat. In the extract below, this kind of transformation is discussed. During the discussion an intuitive appreciation of volume emerges through the comparison of this series of solids. Initially, the children believed that the volume remained the same.

> *[the children have experienced a small variation in the height of the orthogonal parallelepiped]*
> *Christos: Its shape changes but not its volume*

Researcher: I go on here.[reducing the height]
Children: No.
Christos: The same water we can put inside
Researcher: What does it happen now? [when it becomes flat]
Children: It will disappear.
Researcher: It will become flat.
Teacher: So, what has happened to the volume?
Children: [for the first time] it becomes smaller.
Teacher: What about a slight change?
Children: It becomes smaller too.

Episode 4

Although the tasks in this episode are of the same nature as in the previous ones, the computer based materials allowed children to move from physical to more mental actions. The tasks become more imaginary and involve an anticipation of actions as the process of transformation needs to be specified more accurately. "Seeing" loses its meaning because of the 2-D representation of the solids. Moreover, the teacher as it has been often reported in the literature in such environments leaves his leading role and he is more open to experiment with the children, This also happens to be the case in this experiment. The children used the possibilities of the environment to measure the length of the edges of the generated solids. In particular, in the following extract they have made two different cubes from an orthogonal parallelepiped, one of side 5cm and one of 2.5cm. The discussion is about comparing the two cubes in terms of their volume. In this extract children's understanding of volume seems to become more analytic than intuitive.

Stefanos: it will be very small [meaning the 2.5 cm cube]
Panayiotis: This is half of mine [meaning the 5 cm cube]
Teacher: Panagiotis said that you made the half.
Stefanos: No.
Panayiotis: I simply say..
Researcher: What?
Panayiotis: Its edges are half.
Researcher: Is it the half cube?
[Some children say no, others yes]
Stefanos: It is the fourth.

Concluding remarks

Analyzing how this dynamic environment is developed in the classroom, issues concerning the implementation of the materials, the development of children's thinking, the kind of teacher's and researchers' interventions and the interrelationship of these elements emerged. Tasks were developed emphasizing the variance and the invariance of the solids' properties. These tasks were based on the dynamic possibilities of the materials, on children's responses and on teacher's and researchers current goals and decisions. Often, teacher's and researchers' goals were in harmony where the common goal was the investigation and development of children's thinking. However in some cases their goals were in conflict. In that moments, teacher's focus was on achieving the goals of the curriculum. Through the negotiation of the meanings that both teacher and researchers attributed to their role we recognize moments where the teacher acts as a researcher and researchers act as teachers who want to see children's learning to take place. This cooperation supported an environment where the focus was on children's mathematical development. Concerning the three dimensional geometrical solids, this study indicates that the children working in this dynamic environment they build a relational understanding which develops from intuitive to more analytic.

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STATES OF UNDERSTANDING RELATIONS AMONG CONCEPTS OF GEOMETRIC FIGURES: CONSIDERED FROM THE ASPECT OF CONCEPT IMAGE AND CONCEPT DEFINITION

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Abstract

In this study eight states of understanding relations among concepts of geometric figures were identified between Level 2 and Level 3 of van Hiele theoretically. These states were assigned to students practically and their validity was clarified through the results of a survey. By using states of understanding considered from the aspect of concept image and concept definition, we can devise material and methods to improve a student's understanding of concepts of geometric figures appropriate for individual students.

Introduction

Abstracting concepts of geometric figures from shapes in daily life is necessary for us to grasp them appropriately. In particular, if we understand inclusion relations among concepts of geometric figures, we can represent definitions or theorems concerning these concepts concisely, as well as, promote problem solving, etc. (de Villiers, 1994). Thus, acquiring an understanding of the concepts of geometric figures is important. However, most students don't understand these concepts appropriately, and in Japan, in particular, inclusion relations among concepts of geometric figures pose a challenge to students (Koseki, 1992; Matsuo, 1992, etc.).

There is a lot of research being done in Japan from various aspects to promote the understanding of concepts of geometric figures. However, few research areas give attention to a system of teaching figures, or propose methods of teaching that consider developmental stages. In other countries, although there are many areas of research on the learning and teaching of geometric figures, many problems have not yet been solved. Van Hiele constructed a theory of geometric thought and demonstrated that students move sequentially from an initial level to higher levels with appropriate instruction and assistance of teachers (van Hiele, 1984). But it is difficult to specify the understanding of students who stay between levels (Burger et al., 1986; Fuys et al., 1985, etc.). In the van Hiele theory, the thinking process from kindergarten to college is divided globally into five levels. There is a wide gap between two sequential levels, particularly between Levels 2 and 3; the conceptual strategies for these two levels are insufficient to promote a transition to higher levels over several lessons on individual concepts of geometric figures.

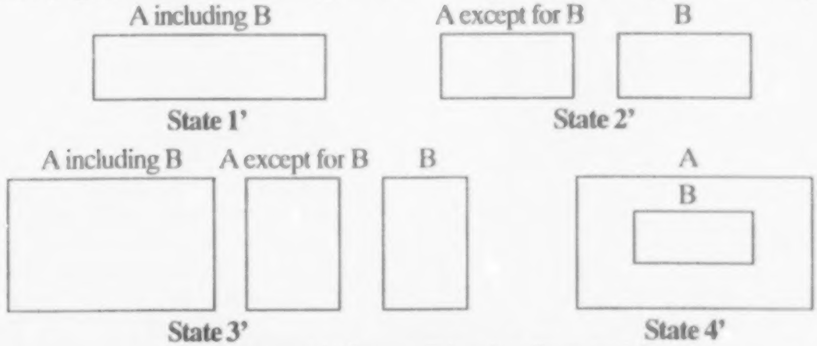
In fact, it has been shown that there are four different states of understanding relations among concepts of geometric figures between Levels 2 and 3 of the van Hiele theory, based on the results of a survey of eighth grade students (Matsuo, 1993), but the theoretical basis has not been clarified. Pegg (1999) identified three categories between Level 1 and Level 2. The purpose of this study is to further clarify the ordered states of understanding for relations among concepts of geometric figures, between Level 2 and Level 3 of van Hiele considered from the aspect of concept image and concept definition.

Theoretical framework

States of understanding and the order of the states. This study focuses on the classification of geometric figures. Four states of understanding relations among concepts of geometric figures in the process of concept formation were identified and ordered (Matsuo, 1999). These states of understanding were constructed on the basis of three stages in the process of development of classification (Inhelder et al., 1958). These states are: students are unable to distinguish between two concepts of geometric figures at State 1, but they are able to identify both concepts respectively at State 2, and distinguish between them based on their differences and regard them as the same based on their similarities at State 3. Further, they are able to understand the inclusion relation between the two concepts at State 4. In this process of concept formation for geometric figures, common features are abstracted and summarized repeatedly (Shimonaka, 1983). Thus, State 1 through 4 are ordered based on the increased number of points required to abstract and summarize common features, that is, shapes, properties and relations of properties of geometric figures (Matsuo, 1999).

Students don't distinguish between two concepts of geometric figures at State 1; instead they group them together. As such, they are considered able to partially recognize a geometric figure by its sides or angles etc. It follows from this that State 1 corresponds to the beginning of Level 2 of the van Hiele theory. At State 4, students recognize the general-specific relation between two concepts of geometric figures. As such, State 4 corresponds to the beginning of Level 3 of the van Hiele theory. Therefore, the transition from State 1 to State 4 identified in this study is equivalent to the process of advancing from Level 2 to Level 3 of the van Hiele theory.

Consideration of states based on concept image and definition. In this study, concept image and concept definition are used in order to further clarify the states of understanding relations among concepts of geometric figures. *Concept image* is a description of a set of properties together with the mental picture of a concept; *concept definition* is a verbal definition that accurately explains a concept in a non-circular way (Vinner, 1983). From the aspect of concept image, the states of understanding are considered based on the relationship between two classified groups of drawings that are assumed to be consistent with a student's concept image of two geometric figures respectively. These states are State 1', State 2', State 3' and State 4', and they are represented (Fig.1) and judged according to the following relations among the classified groups.



(A, B: geometric figures. B is subordinate to A.)

Fig.1: Representation of states considered from the aspect of concept image

These states, State 1, State 2, State 3 and State 4, considered from the aspect of concept image, are illustrated in the example of two geometric figures, A and B, where B is subordinate to A. If a student cannot distinguish A from B at State 1, he/she makes a group consisting of drawings of A, which contain drawings of B at State 1’. When a student can identify A except for B, and B at State 2, he/she makes two groups of drawings, of A except for B, and B, respectively at State 2’. When a student can distinguish A except for B, from B and regard them as the same at State 3, he/she makes two groups of drawings, of A except for B, and B respectively, and a third group which contains both groups at State 3’. Since a student can understand the general-specific relation between A and B at State 4, he/she makes a group of drawings of B and a group of drawings of A which contains the first group, at State 4’.

On the other hand, from the aspect of concept definition, the states of understanding are considered based on the concept definitions of two concepts and the relation between them. Here, a student’s explanation about the two concepts of geometric figures is assumed to be his/her concept definition of the figures. In the process of concept formation, similar points between the two concepts are abstracted as common characteristics and different points between them are ignored repeatedly, thus (1) similar points and (2) different points between them and (3) general-specific relations are important when considering assessing students’ states of understanding.

Judging from the above, five states of understanding considered from the aspect of concept definition are identifiable. That is, an explanation of the concepts of the two geometric figures does not exist (State a); the explanation is about their similarities (State b) or their differences (State c); the explanation is about the similarities and differences between them (State d); and the explanation describes one of two concepts is subordinate to the other (State e).

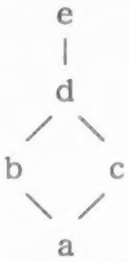


Fig.2: Ordering states considered from the aspect of concept definition

State a, State b, State c, State d and State e are ordered as follows (Fig.2). A student cannot explain either the similarities or the differences between the two concepts of geometric figures (State a), but becomes able to explain either the similarities or differences between them (States b, c), then becomes able to explain both (State d). Finally, he/she can explain the class inclusion between them (State e). By combining State 1’, State 2’, State 3’, State 4’ with State a, State b, State c, State d, State e, a detailed description of which state among the states of understanding relations between two concepts of geometric figures is identified and assigned to students.

From the point of view of concept image and concept definition, a student’s state of understanding of the relations between two concepts of geometric figures is considered in detail. Therefore, states of understanding are specified to students by combining states assigned from the aspect of concept image with states assigned from the aspect of concept definition. These detailed states are decided on based on the states assigned from the aspect of concept image first because according to Denis (1989) imagery appears before verbal representation. The states considered from the aspect of concept image (States 1’ to State 4’) are combined with the states considered from the aspect of concept definition (States a to State e), except for those states that are inappropriate with the former states.

In the first case, a student may make one group of drawings concerning two concepts of geometric figures at State 1'. It is combined with State a, at which he/she cannot explain them verbally, or State b, at which he/she can explain the similarities between the two concepts of figures. Second, a student makes two groups of drawings concerning the differences between two concepts of geometric figures at State 2'. As he/she distinguishes between the two concepts of geometric figures, his/her explanation of these concepts is about only the differences, or about the similarities added to the differences. Therefore, State 2' can be combined with State a, State c, or State d. Third, at State 3' a student makes one group of drawings concerning similarities between two concepts, in addition to two groups of drawings concerning their differences which he/she have made at State 2'. As he/she understands the characteristics of two geometric figures based on their similarities and differences, his/her explanation of these concepts is about only the similarities, or the differences added to the similarities. Therefore, State 3' can be combined with State b or State d. Finally, a student makes a group of drawings that satisfies the properties that are similar between two concepts of geometric figures, and a group of drawings concerning the subordinate concept of one of them at State 4'. The former group contains the latter. Since students explain that one is specific to the other, State e is regarded as the state considered from the aspect of concept definition which corresponds to State 4'.

From what has been said above, eight detailed states of understanding, that is State 1'a, State 1'b, State 2'a, State 2'c, State 2'd, State 3'b, State 3'd and State 4'e are identified. These states are ordered on the grounds of states assigned from the aspect of concept image; and State 4'e is considered the highest one. However, students do not necessarily come to understand inclusions among concepts of geometric figures, State 4'e, as they proceed through all eight states (Matsuo, 1996). As each individual student is assigned an initial different state of understanding for each specified pair of geometric figures, he/she is not assigned the same state of understanding for all pairs of geometric figures. However, their different states for several pairs of figures tend to indicate development toward the highest state; therefore it is judged as their assigned state.

Method

The purpose of the survey was to show that students are assigned any one of eight states of understanding of relations among concepts of geometric figures. A twenty-minute written test was administered before and after lessons on congruence of geometric figures in the fifth grade of elementary school, reflection of geometric figures in the sixth grade, and parallelograms in the second grade of lower secondary school. Two classes each of fifth and sixth grade students in S elementary school and second grade students in H lower secondary school in Saitama, participated in this survey. The reasons why these subjects were selected are as follows. In Japan, fourth grade students at elementary school learn basic quadrilaterals. Generally, fifth and sixth grade students at elementary school begin to think abstractly. Second grade students at lower secondary school understand concepts of quadrilaterals and relations among them by mathematical inference. Students were asked to classify drawings of the geometric figures used in Burger & Shaughnessy (1986), and write the names of the groups classified and the reason why they classified them as they did. In order to clarify students' answers on the written test, an interview was administered with several selected students.

The results of the survey were analyzed as follows. For two groups of drawings made by a student on the written test, a state of understanding was assigned from the aspect of concept image. If the group of drawings made by a student was considered consistent with his/her concept image of a geometric figure, a student was assigned either State 1', State 2', State 3' or State 4', based on the relations between two groups. The criteria used to assign a state to a student were as follows. Here, the concept of B is subordinate to the concept of A. At State 1', it is judged that a student makes one group containing drawings of A except for B, and B. At State 2', it is judged that a student makes two groups of drawings which are consistent with the concept image of A except for B, and B respectively, and there are no common drawings in both groups. At State 3', it is judged that a student makes two groups of drawings that are consistent with the concept image of A except for B, and B respectively, and one group that puts the groups together. At State 4', it is judged that a student makes two groups of drawings which are consistent with the concept image of A and B respectively and the former is a part of the latter.

On the other hand, based on the relationship between a student's concept definitions of two geometric figures, his/her state is assigned with regard to the aspect of concept definition. As concept definition is a verbal explanation about a group of drawings made by a student about a geometric figure, the criteria to assign a state are as follows. At State a, it is judged that there is at most one of the concept definitions for the two geometric figures. At State b, it is judged that the concept definitions of the two figures are the same. At State c, it is judged that there are no common points between the concept definitions of the two figures. At State d, it is judged that there are common and different points between the concept definitions of the two figures. At State e, it is judged that the concept definition of relations between two geometric figures means the general-specific relation. According to the above criteria, a student was assigned a detailed state of understanding relations among concepts of geometric figures.

Results

A student was assigned four states of understanding for each of the relations of squares & rectangles, squares & rhombi, rectangles & parallelograms, rhombi & parallelograms respectively. The highest state assigned for all of the pairs was regarded as his/her state. However, some students were not assigned any one of the above eight states. For example, they made groups, which did not contain any drawings that corresponded to a geometric figure, or that contained drawings unrelated with the figures, or they made different groups from the groups corresponding to the four states. These cases were distinguished from the above eight states. Accordingly, Table 1 shows the states of understanding that the students were assigned.

As Table 1 indicates, there were no students assigned State 3'd, or State 4'e before lessons in the fifth grade; on the other hand, there were students assigned State 3'd after lessons in the fifth grade. Moreover, about 20% of the second-grade lower secondary school students were assigned State 4'e after lessons at that grade. In considering whether to assign students any of eight states or not, it is valid to say that students were able to be assigned any one of the eight states: the pre- and post-test for the fifth grade in elementary school are $\chi^2=48.600(p<.01)$, $\chi^2=45.067(p<.01)$ respectively; the pre- and post-test for the sixth grade are $\chi^2=39.361(p<.01)$, $\chi^2=39.361(p<.01)$ respectively; the pre- and post-test for the second grade in lower secondary school are χ^2 =

20.082(p<.01), χ^2=30.311(p<.01) respectively.

Table 1: The results of assigning states of understanding to students (%)

Grade	State Test	1'a	1'b	2'a	2'c	2'd	3'b	3'd	4'e	Others
Fifth	Pre-	5.0	5.0	26.7	31.7	20.0	6.6	0	0	5.0
(n=60)	Post-	3.3	10.0	33.3	21.7	16.7	5.0	3.3	0	6.7
Sixth	Pre-	3.3	4.9	44.3	13.1	16.4	6.6	1.6	0	9.8
(n=61)	Post-	11.5	8.2	26.2	13.1	14.8	8.2	8.2	0	9.8
Second	Pre-	4.9	9.8	16.4	16.4	4.9	6.6	19.7	0	21.3
(n=61)	Post-	3.3	9.8	9.8	14.8	11.5	8.2	6.5	21.3	14.8

To begin with, before fifth grade lessons the percentage of students who were assigned State 2'a, State 2'c and State 2'd was about 80%. At this grade level, few students were assigned State 3'b, and no students were assigned State 3'd or State 4'e. On the other hand, after fifth grade lessons, we can not say that students' understanding developed as a whole because the number of students assigned State 1'b or State 2'a increased and the number of students assigned State 2'c or State 2'd decreased. However, it is likely that the fifth grade students' understanding improved locally because there began to appear students assigned State 3'd.

Next, before sixth grade lessons, like the fifth grade, the percentage of students who were assigned State 2'a, State 2'c and State 2'd was more than 70 %. In particular about 44% of the students were assigned State 2'a. These students could distinguish two concepts of geometric figures, but they couldn't explain why. On the other hand, after sixth grade lessons, the number of students who were assigned State 1'a and State 1'b increased, but the number of students who were assigned State 2'a decreased dramatically and the number of students who were assigned State 3'd increased. Thus it is clear that students' understanding developed locally.

Finally, before second grade lessons in lower secondary school, most students were assigned each of the seven states except for State 4'e dispersedly, thus the range of states that were assigned students was extensive. We can recognize a local improvement in students' understanding because about 20 % of the students were assigned State 3'd. However, about 20 % of the second grade students were not assigned any one of the above eight states. The reason for this may be that students' answers were diversified because of an increase of mathematical knowledge and in addition, they might not have been able to acquire the necessary learning in elementary school. On the other hand, after second grade lessons, the number of students who are not assigned states except for State 2'd, State 3'b and State 4'e did not increase, and about 20 % of the students were assigned State 4'e. Therefore, it is obvious that students' understanding developed.

Discussion

In this study, the eight states of understanding relations were identified theoretically, and assigned to students practically, clarifying their validity. By assigning states to students, the condition of students' understanding of concepts of geometric figures was clarified in a number of ways, for example, the diversity of students' understanding, how students' understanding develops, etc.

There are three significant points in assigning one of the above eight states to a student and they are as follows. The first significant point is the recognition of the condition of a student's understanding from the aspect of concept image and concept definition. We have two ways of thinking: one is based on images and the other is verbal. These ways of thinking work together to help us think smoothly. In particular visual imagery and language play important roles in understanding concepts of geometric figures. As a concept of a geometric figure is abstracted from shape, we need to consider it through the medium representing it visually. However, visual representation may contain inappropriate information or insufficient information for understanding the concept. Thus we need to hear a verbal explanation of it. From this description, we understand that thinking based on both visual images and verbal explanations is needed in order to understand concepts of geometric figures. It is obvious then, that we can grasp students' understanding of concepts of geometric figures in depth by using concept images and concept definitions based on these two ways of thinking.

The second significance is that we can point out the factors that promote the understanding of concepts of geometric figures. Generally, concept image and concept definition reinforce each other and are in agreement. But a concept image does not always agree with a concept definition. When one's concept image of a geometric figure is consistent with one's concept definition, it is considered that the concept is understood. Therefore, we can discover the factors that promote understanding of the concepts by focusing on the changes in concept image and definition.

To put it more concretely, there are three types of transitions between states; a change of state assigned from the aspect of concept image, a change of state assigned from the aspect of concept definition and a change of state assigned from considering both in tandem. There are three ways to point out factors that correspond to the types of transition. First, our visual imagery functions when a given task is concrete (Paivio, 1979). Hence, concrete operational activities can promote a change in the state assigned from aspect of concept image, for example, the transition between State 1'a and State 2'a. Second, a language system is superior for the task equivalent to verbalization, thus it can promote a change of state assigned from the aspect of concept definition, for example, the transition between State 2'c and State 2'd. Third, let us consider a change of state assigned from the aspect of both of concept image and concept definition. Since Denis (1989) described the use of imagery and verbal representation style depends on situations at a higher abstract stage, it is likely that both styles arrive at the same higher stage. Therefore, concrete operational activities and verbalization are useful in order to promote the transition from State 3'd to State 4'e. It follows from what has been said that we can identify the factors that promote transitions between the states by deciding what activities are necessary to change states assigned from the aspect of concept image and concept definition.

The third significance is that we can create ways to improve an individual student's understanding of concepts of geometric figures based on their present state. We need to understand their unique developmental situation in detail, since there are students assigned different kinds of states at the same grade level, and who reach higher states of understanding in different ways. By using the above states, we can see what states students are assigned at what grade and what percentage of students who are assigned a state that is higher or lower than the percentage of students who are assigned other states at a grade. Accordingly, we can make suggestions about what material, teaching method, etc. are appropriate for what grades.

Conclusion

In this study, State 1, State 2, State 3 and State 4 were identified from the point of view of understanding relations between two concepts of geometric figures. These states were further considered from the aspect of concept image and concept definition, that is, State 1'a, State 1'b, State 2'a, State 2'c, State 2'd, State 3'b, State 3'd and State 4'e. This consideration makes it clear that the transition from Level 2 to Level 3 in the van Hiele theory can be recognized in detail. Therefore, we can devise material and methods for improving a student's understanding of concepts of geometric figures that are appropriate for each student by considering the diversity of their understanding. Clarifying in what order some of eight states approach the highest state remains an area for future study.

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SUPPORTING PRESERVICE TEACHERS' UNDERSTANDING OF PLACE VALUE AND MULTIDIGIT ADDITION AND SUBTRACTION

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Abstract

This paper provides an analysis of a teacher development experiment (cf. Simon, in press) designed to support preservice teachers' understandings of place value and multidigit addition and subtraction. The experiment was conducted as part of an elementary mathematics methods course during the spring semester of 1999 in an American University. Analysis of the episodes in this paper will document the emergence of the preservice teachers' understandings as they participated in activities from an instructional sequence designed to support conceptual understanding of both place value and multidigit addition and subtraction. In doing so, we will highlight the mathematical and pedagogical issues that emerged as the preservice teachers worked to further their own understandings.

Introduction

Current research in mathematics education has provided numerous accounts of students' understandings and misunderstandings of place value and their developing algorithms for multidigit addition and subtraction (Bowers, Cobb, McClain, 1999; Fuson, 1990; Hiebert & Wearne, 1992, 1996; Kamii et al, 1994; McClain, Cobb, & Bowers, 1998). These studies highlight the importance of place value in supporting students' subsequent understanding of multidigit addition and subtraction. In a like manner, preservice teachers' understanding of both place value and multidigit addition and subtraction have implications for how they will teach these concepts in their own classes. In order to support their ability to teach for conceptual understanding, it is important that they have a firm grounding in the mathematics that underlies these concepts.

The purpose of this paper is to provide an analysis of a teacher development experiment (cf. Simon, in press) designed to support preservice teachers' understandings of place value and multidigit addition and subtraction. The teaching experiment was conducted as part of an elementary mathematics methods course during the spring semester of 1999. The course was composed of twenty-eight class sessions. The unit on place value and multidigit addition and subtraction that is the focus of this paper was conducted in five of these sessions spanning a three-week period. The instructional activities utilized during the teacher development experiment were part of the Candy Factory instructional sequence that was developed for use with third-grade students (for a detailed analysis of the Candy Factory instructional sequence see Bowers, 1996; Bowers, Cobb, and McClain, 1999). The intent of the instructional sequence is to support third graders' construction of increasingly sophisticated conceptions of place value numeration and increasingly efficient algorithms for adding and subtracting three-digit numbers. An overarching goal is to support the development of understanding and computational facility in an integrated manner. The original instructional sequence was modified for use with preservice teachers. Part of the modifications included changing the scenario in the Candy Factory so that candies were packed in rolls containing eight pieces each instead of ten. In addition, each box of candies contained eight rolls instead of ten. The intent was to problematize the mathematics for the preservice teachers so that they might then (1) develop an understanding of the multiplicative relationships within place value and (2) develop their own ways of symbolizing transactions in the Candy Factory dealing with buying/adding and selling/subtracting candies. Had the mathematics been trivial, the need to create ways to symbolize their transactions would not have emerged. It was therefore intended that the preservice teachers' own mathematical development be used as a basis for conversations about how to support their students' development. In this way, the preservice teachers could discuss the parallels between their own developing understandings and the understandings of the students whom they would teach. The goal was then to support shifts in the preservice teachers' ways of thinking about how to teach both place value and multidigit addition and subtraction so that students of the preservice teachers might develop conceptual understanding instead of mere proficiency with meaningless algorithms.

In the following sections of this paper, we begin by describing the theoretical framework that guided the analysis. Next, the data that is the focus of analysis is described. We then outline the instructional sequence that was used in the classroom. Against this background, we provide an analysis of classroom episodes intended to document the preservice teachers' developing understandings of place value and algorithms for multidigit addition and subtraction, and how these concepts might be introduced in elementary classrooms.

Theoretical Framework

The analysis reported in this paper was guided by the emergent perspective (cf. Cobb & Yackel, 1996). The emergent perspective involves coordinating constructivist analyses of individual students' activities and meanings with an analysis of the communal mathematical practices in which they occur. This framework was developed out of attempts to coordinate individual students' mathematical development with social processes in order to account for learning in the social context of the classroom. It therefore places the students' and teacher's activity in social context by explicitly coordinating sociological and psychological perspectives. The psychological perspective is constructivist and treats mathematical development as a process of self-organization in which the learner reorganizes his or her activity in an attempt to achieve purposes or goals. The sociological perspective is interactionist and views communication as a process of mutual adaptation wherein individuals negotiate mathematical meaning. From this perspective, learning is characterized as the personal reconstruction of societal means and models through negotiation in interaction. Together, the two perspectives treat mathematical learning as both a process of active individual construction and a process of enculturation into the mathematical practices of wider society. Individual and collective processes are viewed as reflexively related in that one does not exist without the other. Together, these two aspects provide a means for accounting for the students' activity in the social context of the classroom.

Data

The data for the analysis were collected in an elementary mathematics preservice methods course in January and February of 1999 at an American University. Six class sessions were devoted to a unit on place value and multidigit addition and subtraction. Twenty-three preservice teachers were enrolled in the class. The teacher of the methods course is also the first author of the paper. Data consist of videotapes of each of the six class sessions, copies of all the students' work from assignments completed both during class and outside of class, and the teacher's reflective journal.

Instructional Sequence

The instructional sequence used in the teacher development experiment centered around the scenario of a candy factory that initially involved Unifix cubes as substitutes for candies, and later involved the development of ways of recording transactions in the factory (cf. Bowers, Cobb, McClain, 1999; Bowers, 1996). During initial whole-class discussions, the students and teacher negotiated the convention that single pieces of candy were packed into rolls of eight, and eight rolls were packed into boxes. Ensuing activities included estimating and quantifying tasks designed to support the development of enumeration strategies. These activities involved showing students drawings of rolls and pieces using an overhead projector and asking them to determine how many candies there were in all. In addition, students were shown rectangular arrays of individual candies and asked to estimate how many rolls could be made from the candies shown.

To help students develop a rationale for these activities, the teacher explained that the factory manager liked his candies packed so that he could quickly tell how many candies were in the factory storeroom. In order to record their packing and unpacking activity, the students developed drawings and other means of symbolizing as models of their mathematical reasoning. The goal of subsequent instructional activities was then to support the students' efforts to mathematize their recorded packing and unpacking activity. In the next phase of the sequence, students were asked to determine *different ways* that a given amount of candies might be found in the storeroom if the workers were in the process of packing them. For example, 43 candies might be completely packed up into 5 rolls, and 3 pieces, or they might be found as 4 rolls and 11 pieces.

In the final phase of the sequence, the teacher introduced addition and subtraction tasks. These problems were posed in the context of the Candy Factory filling orders by taking candies from the storeroom and sending them to shops, or by increasing the inventory as workers made more candies. The different ways in which students conceptualized and symbolized these transactions gave rise to discussions that focused on their emerging understandings of the addition and subtraction algorithms.

Results of Analysis

Analysis of the data indicate that the preservice teachers' initial understandings of both place value and multidigit addition and subtraction were very superficial and grounded in rules for manipulating algorithms. In addition, they held strong beliefs about the importance of students being able to manipulate algorithms quickly and efficiently, especially on "timed" tests. However, as the preservice teachers engaged in tasks from the Candy Factory instructional sequence, they began to develop an understanding of the importance of a shift in focus from manipulation of symbols to an understanding of the mathematics.

As an example, consider an episode that occurred on the second day of the instructional sequence. The preservice teachers were asked to determine how many total pieces of candy they would have in the storeroom if they had one box, three rolls, and two pieces of candy. In solving the task, the preservice teachers offered a variety of solutions. The first involved drawing pictures of all the pieces of candy including the 64 pieces in the box and the eight pieces in each of the three rolls. In discussing her solution, Brenda acknowledged that this was a tedious approach and stated that in subsequent tasks she would use the fact that a box contained 64 candies and a roll contained 8 candies. However, she noted that she would still need to draw "empty" boxes and rolls so she could "keep track." As a result, Brenda devised a symbol system for these tasks, an example of which is shown in Figure 1.

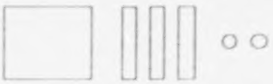

Figure 1. Drawing showing 1 box, 3 rolls, and 2 pieces.

Other students devised notational schemes that supported their ability to determine the total number of candies in the collection by building on the multiplicative structure in the boxes, rolls, and pieces (see Figure 2).

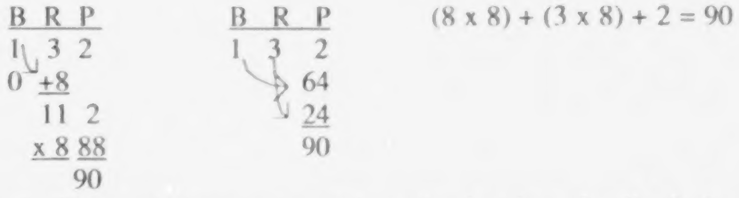

Figure 2. Three solutions to the total candies in 1 box, 3 rolls, 2 pieces.

It is important to note that the preservice teachers developed these schemes in order to make a record of their own activity as they solved the tasks. They were working to devise ways to communicate their thinking. In doing so, they began to develop an understanding of the multiplicative structure that underlies the place value system. This occurred as they began to reason about a box being composed of eight rolls or 64 pieces, and rolls being composed of eight pieces. As they worked to devise *efficient* ways to record their activity, there was an evolution in their schemes that appeared to parallel their developing understandings. This was evident in the shift that occurred from drawing the boxes and rolls to numerical schemes that represented these quantities and the relationships across them.

It is however important to clarify that the preservice teachers experienced difficulties as they developed these ways of reasoning in the candy factory scenario. They were initially frustrated at their inability to solve the tasks quickly. One student commented that although she was having difficulty with the task, she refused to draw pictures. She stated: "I'm a college student and I should know how to do this in my head, but I couldn't." A second student noted, "I had to multiply eight times eight each time to find out how many in a box. You would think I could just remember or look back but I didn't." However, as the sequence progressed, the preservice teachers began to make sense of the mathematics as evidenced by Brad's comment, "I never understood place value, but this is starting to make sense."

After several tasks involving determining the amount of candies in the storeroom, the teacher introduced tasks that involved finding different ways that given amounts of candies could be in the storeroom. These were followed by tasks in which the students had to match inventory forms by determining which pairs of forms represented the same amount of candies. The final series of tasks involved transactions such as selling/subtracting and buying/adding candies. As the preservice teachers worked on these tasks, they developed non-traditional algorithms for addition and subtraction to symbolize their activity. In discussing their methods in whole-class setting, the preservice teachers wrestled with the advantages and disadvantages of letting algorithms emerge from the students' activity. They believed that the traditional algorithms should be the goal of instruction. Allowing students to devise their own methods for adding and subtracting did not fit with their current beliefs about what it means to teach mathematics. In addition, many of the preservice teachers were unsure of the validity of the alternative methods. Their superficial understanding of the mathematics was insufficient for them to be able to judge the mathematical basis for these methods. Situations that emerged such as subtracting by starting from the left provided opportunities for the preservice teachers to not only further their own understandings of the mathematics involved in the algorithms for multidigit addition and subtraction, but to also question their beliefs about their role as a teacher in supporting their students' understanding of traditional algorithms.

As an example, consider an episode that occurred on the last day of the instructional sequence. The preservice teachers had worked several problems outside of class that involved buying and selling candies. The first task discussed in whole-class setting asked: *There are 2 rolls and 5 pieces of candy in the storeroom. The workers make 4 rolls and 6 pieces. How many candies are in the storeroom?* As students shared their ways of solving the task, it became evident that they found it useful to first find the total number of pieces and record that amount *before* considering if the pieces could be packed into a roll. This can be seen in the preservice teachers' solutions shown in Figure 3.

$$\begin{array}{ccc} \mathrm{B} & \mathrm{R} & \mathrm{P} \\ \hline & 2 & 5 \\ & \underline{4} & \underline{6} \\ & 6 & 11 \\ & 7 & 3 \end{array} \qquad \begin{array}{ccc} \mathrm{B} & \mathrm{R} & \mathrm{P} \\ \hline & 2 & 5 \\ & \underline{4} & \underline{6} \\ & 6 & 11 \\ & \underline{+1} & \underline{-8} \\ & 7 & 3 \end{array} \qquad \begin{array}{l} 2\,\mathrm{R}\ 5\,\mathrm{P} \\ \underline{4\,\mathrm{R}\ 6\,\mathrm{P}} \\ 6\,\mathrm{R}\ 11\,\mathrm{P} = 7\mathrm{R}\ 3\,\mathrm{P} \end{array}$$

Figure 3. Solutions to the task.

After the three solutions in Figure 3 were on the board, the teacher posed the following question.

> Teacher: I want you to notice that in each case you put a two-digit number in the column for pieces. Is that okay? Could you do that if we were doing the algorithm?

The subsequent discussion then focused on the preservice teachers' ways of solving the task and the relationship between their notation and the conventional algorithm. Several of the preservice teachers stated that it was important for them to keep a record of all the pieces before they thought about "packing up." They related this to their understanding of the algorithm for addition and noted that you could do the same thing as long as you could devise a way to "keep track." Other preservice teachers then offered that they really did not understand what the carrying notation in the traditional algorithm signified, but acknowledged that it "gave you the right answer." At this point, the teacher used the problem being discussed as a way to investigate the relation between the preservice teachers' ways of solving the task and the more formal algorithmic approach. In doing so, the preservice teachers came to understand the significance of the notation. They realized that the "one" that was carried really signified one ten. They also noted that the notation was just as important to understand when subtracting.

Following this discussion, the teacher selected a subtraction task to be discussed in whole-class setting: *There are 2 boxes, 5 rolls, and 3 pieces of candy in the storeroom. If you send out 4 rolls and 5 pieces, how many candies are left in the storeroom.* The first preservice teacher to share her solution noted that she had started working from the left, but then realized she did not have enough pieces. As a result, she began again, this time working from the right. She stated, "I realized I had to get more pieces." She then worked the problem on the board as shown in Figure 4.

$$\begin{array}{r} 3\,\mathrm{B}\ \overset{5}{\cancel{6}}\,\mathrm{R}\ \overset{10}{\cancel{2}}\,\mathrm{P} \\ \underline{-\qquad 4\,\mathrm{R}\ 5\,\mathrm{P}} \\ 3\,\mathrm{B}\ 1\,\mathrm{R}\ 5\,\mathrm{P} \end{array}$$

Figure 4. Solution to transaction task.

At this point, the teacher asked the other preservice teachers if they could solve this task by starting with the boxes. In the discussion that followed, several of the preservice teachers argued that you could still unpack a roll to get more pieces, even if you started with the boxes. They then discussed how that could be notated to clarify what had happened in the storeroom. In doing so, they devised the notation shown in Figure 5.

$$\begin{array}{ccccccc}
 & & & & & 10 & \\
 & 3 & B & 6 & R & \cancel{2} & P \\
- & & & 4 & R & 5 & P \\ \hline
 & 3 & B & \cancel{2} & R & 5 & P \\
 & & & 1 & & &
\end{array}$$

Figure 5. Notation resulting from starting with boxes.

Discussion then focused on the mathematical accuracy of each way and the similarities and differences of each approach. In the process, the preservice teachers also noted the importance of the notation fitting with the students' activity so that it "will make sense and not be just a set of rules." Not only were they working to make sense of the mathematics for themselves, but they were also expressing a change in belief about the role of algorithms in the mathematics classroom. Earlier we saw evidence of their belief in a correct procedure to use in solving the problem. In this discussion, however, they are acknowledging the importance of building from students' understandings so that the algorithms have meaning in the context of the students' activity.

As a follow-up, the teacher asked the preservice teachers to discuss how they would solve the task: *5 boxes and 2 pieces take away 4 rolls and 6 pieces.* The first solution was offered by Anne who stated:

> Anne: First I unpacked a box then I had to unpack a roll. So I sent out my pieces and that left four then three rolls and four boxes.

As Anne explained, the teacher notated as shown in Figure 6.

$$\begin{array}{cccccc}
 & & 7 & & & \\
4 & & \cancel{8} & & 10 & \\
\cancel{5} & B & 0 & R & 2 & \cancel{P} \\
 & & 4 & R & 6 & P \\ \hline
4 & B & 3 & R & 4 & P
\end{array}$$

Figure 6. Notating Anne's solution.

After she finished, Terri asked, "Why did she put eight then seven on the rolls?" This question prompted a discussion about how to notate unpacking, or borrowing across a zero. The preservice teachers engaged in a lively discussion about how to subtract when "you don't have enough." Although many of the preservice teachers understood the procedure as explained by Anne, others commented that they had never understood what to do "when you have a zero." At this point, Kathryn offered that she knew that in the algorithm "when you have a zero, you put a ten and then a nine." However, she continued by stating that she could never remember which came first. For her, these were meaningless rules in a procedure that were intended to produce a correct answer. They did not involve decomposing quantities in order to be able to perform a calculation. At the end of the discussion Kathryn commented, "This is the first time I have ever understood *why* you do that."

Conclusion

Although the instructional tasks used in the teacher development experiment in the methods class were modified from those used in a third-grade classroom teaching experiment, it appears that the underlying learning trajectory that guided the third-grade students' development can also support preservice teachers' mathematical development. This is an important finding in that it offers a way for methods instructors to concurrently support both the mathematical and pedagogical development of their students. In doing so, the preservice teachers can think about the mathematics from the perspective of a student as they wrestle with the problems, and then "change hats" and step back from their own activity and reflect on how these materials could be used to support the mathematical development of *their* students. Not only is this important from a pedagogical point of view, but it also provides opportunities for shifts to occur such that the discussions are then grounded in the mathematics that will be taught.

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Spatial Abilities in Primary Schools

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Abstract: *We will analyse the development of spatial abilities in primary grades. After teaching a curriculum unit in 8 primary school classes in Germany we administered a test (third graders, n = 184, control group n = 168). But the test results, though showing a significant improvement in spatial abilities, were not satisfying: Did the test really measure an advancement of cognitive processes? Our analysis of cognitive processes during spatial problem solving suggests to understand "spatial abilities" as comprising visuo-spatial mental operations as well as abilities related to specific VAN HIELE levels, proceptual flexibility (TALL) and communicational abilities (MEISSNER).*

1 The curriculum unit

There is not much geometry teaching in German primary schools. To further the students' knowledge in geometry we have developed a unit which can be taught in grades 2-4, independently from other geometry curriculum units. It is based on the ideas of constructivism: Each lesson includes activities which the pupils design on their own. The pupils themselves "invent" a new concept by acting and manipulating along very general guidelines. And after inventing a new "concept" or "theory" they get a lot of examples/counterexamples for testing and/or reorganizing their thesis.

Since daily life is full of geometric experiences, and these are generally three-dimensional, we, in this unit, concentrate on spatial geometry. The unit's main material to work with is a set of about 35 solids, made from styroform, wood, paper developments, or plastics. In seven lessons the pupils connect these solids with real life situations, they discover geometrical properties, they construct developments of pyramids, of rectangular solids, of houses, and finally they build a complete little village (for more details see MEISSNER 1995).

2 Evaluation

Before and after teaching the unit in autumn 1998 to 8 third grade classes of primary schools a test was administered to measure the pupils' improvement in spatial abilities. For each experimental class there has been a third grade control class in the same school. The test consists of 9 tasks of which 8 stem from known tests on spatial abilities (cf. ELIOT et al. 1983) and one is based on a geometrical activity designed by Besuden for promoting spatial abilities (BESUDEN 1984). Two examples follow below, a third example is shown lateron (Fig. 7).

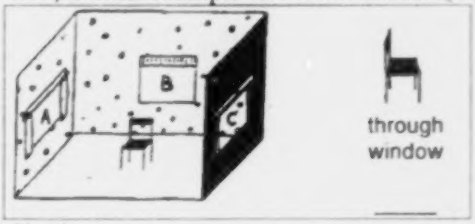


Fig. 1

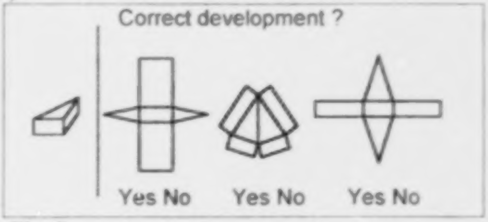
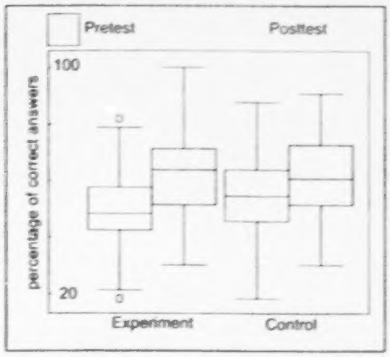


Fig. 2

Descriptive statistics for the pre- and posttest results in the experimental and the control group are as follows:

		Mean	Standard Deviation
Experiment	Pretest	.4972	.1269
(N = 184)	Posttest	.6100	.1410
Control	Pretest	.5328	.1800
(N = 168)	Posttest	.6142	.1337

Fig. 3



Both groups show an improved performance in posttest compared with pretest results. Part of the improvement must be explained, as the control group shows, as test sophistication (Lienert 1994, 124). However, repeated measures analysis of variance (ANOVA) reveales a main effect of the group affiliation factor on the test improvement [F(1, 350) = 6.720, p = 0.010]. The test data confirm that the teaching unit has a highly significant effect on the improvement in spatial abilities. (Different pretest results of the experiment and the control group do not affect the suitability of ANOVA.)

These test results were not satisfying for us in comparison to our personal observations from studying the teaching-learning processes (each lesson was video-taped, we have about 2400 minutes to evaluate). We had expected a bigger difference between pretest and posttest. Did we really measure an advancement in spatial abilities with a test that we have designed similar to those used in many other psychological areas? We think describing and measuring spatial abilities should consider, for the use in mathematics education, also specific aspects of learning mathematics (cf. BISHOP 1983, M.A. CLEMENTS 1983). But what does that mean?

3 Analysing cognitive processes in solving spatial tasks

3.1 Visuo-spatial mental operations

At the core of most definitions of "spatial ability" are certain mental operations that we will refer to as "visuo-spatial mental operations". We understand cognitive processes as "being visual" or "being spatial" when they can be described in visual or spatial terms. Being asked, for example, to tell the number of the doors in the corridor where my office is, I would recall the process of walking through the corridor in a mental repetition of what I do every morning, now with concentrating on the doors as they move along my "inner eye".

When we call mental operations like this as visuo-spatial, we do this by describing them as if they were operations in real space. That there is reason to assume a functional equivalence of mental imagery and visual perception has been indicated in many experiments (cf. FINKE 1986). However, mental images are not exact copies of their real origins. Although my mental image of the corridor looks complete with a door to the men's restroom, it still lacks the door to the ladies' of which I know it must be there. This is not simply a matter of being forgetful. The mental image is the product of a constructional process which is influenced by many other cognitive abilities than just visuo-spatial (cf. D.H. CLEMENTS 1992). What kind of cognitive processes are involved, especially in the context of mathematics education, could be cleared up by an analysis of geometrical problem solving processes. When, furthermore, we apply theories for the learning of mathematics to the analysis we gain a description of spatial abilities that is as we believe more suitable for the use in mathematics education. We will demonstrate this by analysing transcripts of video recordings that each shows a child from our project working on a geometrical task.

3.2 Geometric thought

VAN HIELE (1986) describes five levels in the development of geometrical concepts. When learning about geometric shapes, for example, a child starts with identifying and naming simple shapes as "wholes" (level I). With level II a child has learned to differ between shapes by comparing geometric details. Next, with level III, comes a first insight into the logical ordering of shapes and properties. (The final two levels IV and V are not considered in primary geometry as they deal with deduction and axioms.) In the context of spatial abilities, not being able to identify the geometric properties of an object might result in a less systematic visuo-spatial operation on the object. For illustration consider the following case:

The teacher has introduced the vertices, edges, and sides of a solid, e.g. the edges are to be identified by stroking with the fingertip along the edge of the solid. Now the pupils are asked to count the vertices, edges, and sides of a solid that they are free to choose. Lisa (age 8:8) takes the paper model of a truncated pyramid and counts its edges as follows

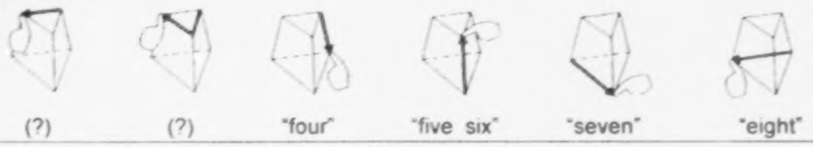


Fig. 4

Lisa's counting is obviously systematic. First she counts the edges of the truncated pyramid's top side. Then she continues with the edges that link the top with the base, of which she indicates only two but counts three. Finally she counts two edges of the pyramid's base. Here she finishes. Speaking with VAN HIELE, Lisa has already left behind level I, where the papermade object would have been perceived just as such. She structures it, i.e. she divides it into top, base and the linking parts. Yet, she still does not identify the parts as geometrical shapes. At least, because she stops after having counted two edges of the base, she obviously does not make use of the fact that the base is a triangle.

For performing a visuo-spatial operation effectively, it is necessary to restrict all visual information to only what is needed, here to disregard the size, or shape as a "whole", or the material of the truncated pyramid and concentrate on certain geometrical properties that are necessary for a systematic counting of its edges. Eventually, this could lead to replacing the visuo-spatial operation altogether by a reasoning that is more of a logical kind (level III): When top and base have been identified as triangles we have already 2×3 edges. The remaining edges connect the two triangles at their vertices, hence together with these 3 we have 9 edges altogether.

3.3 Spatial procepts

The term "procept" has been coined by Tall (1991) to reflect the fact that many mathematical concepts have both a static and a procedural character. For example "3+4" represents the number 7 in the form of a sum. It also represents the process of adding 4 to 3, or the process of counting on four more steps from "3". Hence, it denotes an instruction that two numbers have to be added, and at the same time it stands for the result of an addition. "3+4" represents, therefore, both a concept and a procedure, hence it is named a procept.

The main function of a procept is to facilitate advanced reasoning by encapsulating basic procedures like adding into new objects of mathematical reasoning. While this applies to many concepts in arithmetic, algebra, calculus etc. it is not clear that there are procepts in geometry. In fact, when we understand the progress to an advanced geometrical reasoning as following the levels of van Hiele, where, starting from a holistic perception of shapes, their geometric properties are gradually singled out and eventually become objects of deduction and axiomatic reasoning, Tall is right in saying that advanced geometric concepts do not result from encapsulating procedures (Tall 1991, 254).

But this could be different when we think of the advancement in spatial reasoning. There are levels of spatial reasoning where the visuo-spatial mental operations involved are not the dominant part of reasoning any more. One could say they have been compressed to allow a more effective strategy in solving complex spatial problems. As an example, consider the following situation:

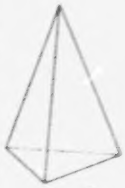


Fig. 5

The teacher shows a model of a three sided pyramid (Fig. 5) and asks the class: "How did the cardboard paper look like before I folded it to make this pyramid?" Friederike (age 8:2) draws a square and adds three triangles to its sides (Fig. 6). She then shows with her hands how to fold the pyramid, points to the side of the square where there is no triangle, and says: "Then there's a hole, isn't it?"

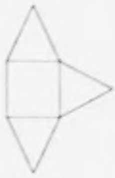


Fig. 6

By looking at Fig. 6 an experienced geometrician sees that it does not represent the development of a solid. He can decide without actually folding the net. In his reasoning the process of folding has been encapsulated to the static concept "development". Friederike however has some notion of "development" in which she still needs to carry out the process of folding explicitly, as her hands indicate. Thus we consider "development" as a procept in spatial reasoning.[1]

With "procepts" we can explain the advancement to a higher level of spatial reasoning that is implied in the progress from procedure to concept. GRAY calls it "proceptual thinking", by which he means a flexibility in using procepts as encapsulations of procedures. In particular, proceptual thinking includes the ability to revise an encapsulated procedure to meet new demands (GRAY 1994, 2). For this we present an example: We have taught children make the net of a pyramid by placing a model onto a sheet of paper and then repeatedly tilting it from its base onto one side and back to the base again, each side being encircled with a pencil. The resulting figure would be a star shaped net. Next, we have asked them to make the net of a rectangular solid. What we have experienced many times is that in strictly following the learned procedure they forget the solid's upper side and produce a net that would fold to an open box. The concept "development" they have acquired so far is based on a procedure of what could be called "tilting from and back to the base". With the rectangular solid this procept must be revised by extending the procedure of "tilting".

4 Communicating on visuo-spatial aspects

A pupil might very well know how to solve a mathematical problem by a certain procedure, and he might also know why this is a reasonable way to solve the problem. But as a novice to the use of mathematical language he might not be able to explain his solution such that the teacher thinks he is correct. Often it happens that a pupil's ability in communicating on a formal level is, by the teacher, not reflected as an extra qualification but is being expected tacitly, especially when for him the formality has already become a natural part of his (professional) language. A relational understanding (cf. SKEMP 1978) with adequate communicative abilities is described by MEISSNER (1992) as "communicable" understanding.

In the context of assessing pupils performance in spatial abilities, where we heavily rely on the observable behaviour as we do not have direct access to their cognitive processes, the communicative abilities become part of the abilities being assessed, especially when, as in the context of spatial abilities, figures are tacitly expected to be the easiest part of a task's instruction. The following example indicates that this is not always the case.

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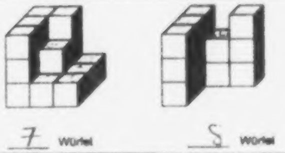
[1] There is a close connection to PIAGET's notion of "symbol" here. In differentiating between three kinds of images of which one merely represents the object itself and another expresses phases of uncoordinated actions upon the object it represents, Piaget describes a third kind of image that, though iconic in appearance, is "entirely concerned with the transformation of the object [...] In short, the image is now no more than a symbol of an operation, an imitative symbol like its precursors, but one which is constantly outpaced by the dynamics of the transformations" (PIAGET 1967, 296). In other words, the net is understood to be symbolic in that it stands for operations which, though not apparent in the figure itself, are associated with it.

In a written task the pupils are asked to tell the number of cubes in a building that is pictured on the sheet. Vanessa's (age 9:2) solution is shown in the figure.
("Würfel" is the German word for cubes)

Fig. 7



It appears that she has counted the squares instead of the cubes. One explanation for this is that she takes the word "cubes" as meaning "squares", which can be observed very often in primary grades. Another explanation would be that Vanessa's communicable understanding of perspective drawings is not sufficient. If she were shown the real (three-dimensional) cube building she would probably be able to identify the number of cubes. Hence, with assessing her spatial abilities with a task like this we, at the same time, assess her ability in communicating visuo-spatial information.

5 Conclusions

5.1 Cognitive aspects

"Spatial abilities" is a term for a broad range of different aspects. In the center we see "visuo-spatial mental operations" which can be described as if they were operations in real space. And there are many tasks to test "spatial abilities" which can be solved by the use of visuo-spatial mental operations. But if such a task is solved correctly it is not necessary that it was done by the use of visuo-spatial mental operations. That means the correct solution of such a task does not necessarily give informations about the ability of using visuo-spatial mental operations. In fact, it appears that more advanced solving strategies may reduce visuo-spatial mental operations.

The same task may be solved quite differently according to the developmental stage of the task solver. The same task measures on a different level different abilities, a correct result in itself is not very informative. I.e. there may be a visuo-spatial mental operation on van Hiele level II and a logical reasoning on van Hiele level III.

In addition there are spatial abilities which seem to develop as procepts. In this case the mental processes in solving a task depend on the developmental stage of that person's procept. A simple question may further the revision of encapsulated procedures. For example, in our project pupils very often constructed their first net of a rectangular solid by applying the tilting strategy of pyramids. Net and solid on the table and the question (pointing on the solid's missing side) "Where is this side?" often produced two reactions, a broad laughter by the children and, at the same time, a double tilting of the solid on the original net to add the missing side.

5.2 Communication problems

"Spatial abilities" are based on the cognitive development of the individuals. But the measurement of "spatial abilities" also depends on the format of "communication". In our project the emphasis of all individual activities is on acting in the three-dimensional space. When we use paper and pencil we often do this to concentrate on (one- or) two-dimensional properties of the three-dimensional objects. I.e. drawing windows and doors on the development of houses is basically a process of folding up the net, realizing three-dimensionality, folding down and painting, folding up again to control, etc.[2]

Many of our test items were presented by perspective drawings of three-dimensional objects (cf. Fig. 7). But in our teaching unit we do not train any drawing of three-dimensional objects as perspective drawings on a paper. How can we be sure then that such a perspective drawing will be understood as we expect? A wrong answer may originate from misinterpreting the drawing (and/or the accompanying verbal or written explanations) and not because of low developed "spatial abilities". With an adequate presentation of the task our pupils perhaps may be able to solve any given problem in the three-dimensional space. If they fail our first question should be whether there was a misunderstanding of the task.

To summarize, correct answers of a test which claims to measure spatial abilities do not necessarily give much insight into the really existing "spatial abilities", and on the other hand, incorrect answers may not originate from a non-existence of "spatial abilities". That means, test results do not necessarily give insight into the underlying cognitive processes for solving the task.

5.3 Alternatives?

At the moment our project group is discussing the possibilities of how to design a new (written) test. A first discussion is on whether we should train the coding and decoding rules for constructing and interpreting perspective drawings of three-dimensional objects. We hesitate because this was not a goal of our unit for primary grades. From other experiments in primary schools we know that children are capable of developing own strategies of representing spatial relations and make themselves understood by their peers (WOLLRING 1998). Shall we develop the next test on that base?

Of course the quality of a teaching unit also can be described by case studies. But the evaluation of this type of research and especially the generalisation of the results is much more sophisticated. We will reflect this in a separate paper.

Another question is how to formulate the tasks or how to incorporate questions into the test to get also insight into the task solving process. On which level did the pupil solve the problem, which were the cognitive processes?

We have some experiences with multiple choice tests in mathematics for university students. Each question can be answered with different strategies and according to the chosen strategy there may be one or more different (correct or incorrect) answers. (The number of correct answers is not known to the students.) The number of distractors for each problem depends on the reasonability, that means on the availability of different (correct or incorrect) strategies (which must be foreseen in advance by the test developer). Can we profit from these experiences for the designing of a new test to analyse spatial abilities? During our presentation at the PME conference we hope to present some first ideas.

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[2] In all our villages at the end of the unit we find buildings with an individual design. The pupils have built the grocery store from the school's neighbourhood, their dwelling house, the church, etc.

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ANALYSIS AND SYNTHESIS OF THE CARTESIAN PRODUCT BY KINDERGARTEN CHILDREN

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Kindergarten children were individually interviewed in this study on tasks that involve cartesian multiplication, where the two multiplied sets are either two different sets (AXB) or identical sets (AXA). Children's understanding of the cartesian product structure was tested by observing the way they construct ordered pairs, classify a given set of pairs, and predict the size of the product-set before constructing it. The interview was dynamic in nature, and enabled children to learn through task experience and with the use of an empty two-dimensional table. The findings show that AXB tasks are easier than AXA, that most children are able to build the AXB product-set, and 33% can even analyze the two dimensional structure of a given AXB product-set.

Although new math curricula stress the importance of discrete mathematics, the tasks that are often suggested involve the construction of quite complicated sets, and make us wonder what understanding is gained by them. This study tries to offer a set of tasks, which can facilitate better understanding of the structure of cartesian multiplication. It investigates the ability of kindergarten children to cope with some tasks, and the effect it has on their understanding.

Less complicated cartesian product construction tasks were used by English (1992), who tested a large range of ages, 4-12. English found six stages in children's search strategies, yet most of the 4-6 children used the most primitive one, a trial and error strategy. A different type of tasks was suggested by Inhelder and Piaget (1964), who asked children to classify pictures or objects according to two features, or complete a two dimensional matrix. They found that these classification tasks were easy for 4-5 year olds, who solved them in perceptual or graphic terms, and not in operational terms.

These studies and several other studies (such as Piaget & Inhelder (1958, 1975), English (1988, 1993, 1996), Maher, Martino & Alston (1993), Maher & Martino (1996), and Sarig-Aharon (1997)) which dealt with cartesian multiplication, involved either construction or classification tasks and not both. Some of these studies used AXB (two different sets) tasks and some used AXA (two identical sets) tasks. Because of these and several other differences in task variables, it is difficult to compare research results. The present study tries to compare the effect of these variables.

This study focuses on kindergarten children's ability to perform a sequence of tasks which include the construction of a product-set, and also the classification of a product-set. It also compares their performance in AXB (two different sets) tasks and AXA (two identical sets) tasks. This study is a part of a larger research, which investigates the development of children's understanding of the cartesian product from kindergarten through ninth grade.

PROCEDURE

The sample included 45 kindergarten children (21 girls and 24 boys) from 5 middle class preschools in Israel. Each child was individually interviewed by the researcher two or three times. The children were randomly assigned to four groups, which differed in the context and the set similarity (AXB or AXA) of the tasks. Each child was given a sequence of 3 tasks:

1. Constructing all possible 'tables' using two Lego blocks: a 'table base' made of one small block (out of two different colors), and a 'table top' made of a bigger block (one out of three different colors):

Following the table construction the child was asked to classify all the tables.

2. Constructing all possible two-part cards by stamping them. The stamps were either chosen from the same set in an AXA task, or from different sets in an AXB task. Following the set construction the child was asked to classify the constructed set.

3. In the third task the child was asked to classify a product-set constructed by the interviewer.

In addition to constructing and classifying children were also asked to predict in different cases the number of elements that would be created before they actually constructed them all.

RESULTS

Children's understanding of the cartesian structure were deduced from their construction methods and from their answers to different questions during the interview (e.g. prediction of final number). Three main principles were assessed:

a. Understanding the basic principles of the task, which included: knowing that a pair is built by taking one and only one element from each of the two sets; understanding that a pair is one element in the new product-set; accepting the fact that each element of the maltiplicand-set can appear in several of the product pairs (unlike the situation in an additive structure); and understanding that each pair should appear only once in the product-set (no repetition in the pair construction). Children's understanding of these principles is recorded in table 1.

Table 1: Percentage of children acquiring the basic product principles.

Principle	Task 1 ('tables') AXB (n=45)	Task 2 AXB (n=24)	Task 2 AXA (n=21)
One element from each set	100% (45)	96% (23)	91% (19)
A pair is a single element	100% (45)	96% (23)	91% (19)
Same element in several pairs	98% (44)	96% (23)	91% (19)
No repetition of pairs	89% (40)	75% (18)	81% (17)

It should be noted that table 1 presents the *final* percentages of children exhibiting knowledge of each of the principles. In some cases children learned the principle in the course of the interview. For example, out of the 98% of the children who knew that the same base or the same top can appear in several pairs, 9% learned it during the table building task.

The following protocol excerpt demonstrates the difficulty in understanding the basic principles of constructing the product-set:

> Dana (5;9) is performing an AXA task. She is given cards that are divided into an upper and lower part, and asked to construct all possible cards by stamping each part with one of 3 different animal stamps.
>
> Dana: (stamps one card with a bird on top and a butterfly underneath.)
> Interviewer: *Are there any other combinations?*
> Dana: *Yes. A horse.*
> Interviewer: *You are asked to make all possible* ***combinations****. So that one animal will be on top and another underneath, whichever you choose, but if you have a butterfly on top and a bird at the bottom, you are not supposed to make another one exactly like it.*
> Dana: (Puts the horse stamp on the top part of a card and stops) *There are no more.*

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b. Having an internal multiplicative structure, which included: predicting the final number of elements in the product-set; using a well planned, odometric pair construction process (each element from one set is systematically paired with every element of the other set); and predicting the size of the product in the following cases : addition of one element to one of the sets, addition of one element to each of the two sets, exchanging the number of elements in the two sets (having two options for the table top and three for the base instead of three and two correspondingly). Table 2 shows children's knowledge of the second group of principles. As mentioned earlier with regard to table 1, here, too, some of the children became better predictors in the course of the interview.

Table 2: Percentage of children having an internal multiplicative

Aspect	Task 1 (tables) AXB (n=45)	Task 2 AXB (n=24)	Task 2 AXA (n=21)
Increase of one set	47% (21)	---	---
Increase of each set	---	13% (3)	5% (1)
Exchange of set sizes	20% (9)	---	---
Final product size	5% (2)	0%	5% (1)
Odometric pair construction	9% (4)	17% (4)	0%

Some insight on the difficulty involved with the AXA cases comes from observing the process of pair construction. It was found that in these cases children tended to associate pairs that had the same two elements in a different order and then to make pairs of identical elements.

c. Children's behavior in the classification tasks is another important source for identifying their conception of the whole product structure. In a typical classification task the child is asked to classify a given set of pairs, e.g. a set of tables, according to his own criterion. The child might choose to sort the tables by the color of the table base, put all tables with a red base together, and the tables with a blue base together. Following the first classification, the child is asked to suggest another one. He might

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suggest to classify the tables by the color of the table top. Once the child has demonstrated these two sorts, the interviewer asks him whether he can figure out a way to organize the tables, so that both kinds of classifications can be seen at the same time. As expected, this is a difficult problem for a young child. At this point the interviewer might suggest the use of an empty two-dimensional table. If that does not help, the interviewer puts the elements of one set (e.g. table tops) above the columns and the elements of the other set (e.g. table bases) by side of the rows. The child is now asked to place each pair in the right cell of the table.

Table 3 depicts children's ability to cope with a classification task at different difficulty levels, in AXB and AXA tasks. The results indicate performance differences that seem to depend on several parameters: Tasks of type AXA are more difficult than AXB. Task 3, in which the child did not build the product-set, was more difficult than task 2, in which the child constructed the pairs before being asked to classify them.

Table 3: Percentage of children at different types of classification tasks and in different task levels.

Level	Task 1 AXB (n=45)	Task 2 AXB (n=24)	Task 2 AXA (n=21)	Task 3 AXB (n=24)	Task 3 AXA (n=21)
Recognizing full product-set	64% (29)	54% (13)	29% (6)	---	---
Sorting in table, margins given	67% (30)	71% (17)	48% (10)	63% (15)	33% (7)
One dimensional organization	62% (28)	71% (17)	33% (7)	58% (14)	24% (5)
Double one dimensional organization	29% (13)	46% (11)	10% (2)	50% (12)	14% (3)
Two dimensional organization	11% (5)	33% (8)	5% (1)	42% (10)	5% (1)
Common structure of tasks	---	17% (4)	0%	8% (2)	10% (2)

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The sorting of the AXA pairs reveals the same internal structure as in the process of constructing AXA product: children tended to treat pairs of identical elements as a separate type of pairs, and also put together pairs that had the same two elements in a different order.

DISCUSSION

This research investigated kindergarten children's understanding of the structure of a cartesian product by examining their ability to construct and to analyse products of two sets.

The results indicate that most of the children are able to construct elements of a cartesian product, or learn to do so in the course of the interview with some guiding, and 54%-64% of them complete all the AXB pairs possible. This is quite surprising in view of the fact that fourth and fifth grade children have been found to have difficulties in constructing all the possibilities in combinatorial problems. It might be the result of the set similarity of the task, and it might also mean that children are having trouble with analysing the problem situation rather than with the mathematical model which involves the cartesian product.

The control of task characteristics: AXB versus AXA tasks, construction versus classification, and classification of a self-constructed product versus a given product (created by the interviewer), made it possible to identify sources of difficulty. The AXA classification tasks were more difficult than the AXB tasks because of children's focus on the relations between elements within pairs. This seems to be a perceptual obstacle rather than a mathematical one, and might therefore suggest that we prefer to use AXB tasks in working with kindergarten children.

Although classification tasks were found to be difficult, they were easier when children had a sequence of tasks starting with product construction and then continuing with product classification of the set created by them. This task sequence apparently created a self-learning opportunity.

In general, we can say that children tended to take advantage of different learning opportunities either guided or unguided. In her efforts to guide children in the course of the interview, the interviewer used an empty table. This table was used as a tool to facilitate, in Vigotsky's (ZPD) spirit, the organization of the product in a two dimensional matrix. Some of the children were able to benefit from this tool and appreciate its power in representing two different classifications at the same time.

Kindergarten children's performance and learning process in this study implies that the cartesian-product structure can be introduced at this age via AXB tasks of construction and classification, and facilitated by tasks that involve the pairing of real objects.

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THE "CONFLICTING" CONCEPTS OF CONTINUITY AND LIMIT - A CONCEPTUAL CHANGE PERSPECTIVE

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Abstract

The theories of conceptual change analyze the prior knowledge of the students in order to explain learning difficulties. In this report we use those theories to explain the difficulties students have in understanding the definition of the concept of continuity of function. The students from upper secondary schools calculus classes described in their own words how they understood the concepts of continuity and limit. According to the results students seem to be very prone to rely on their initial presumptions and everyday experiences. It is suggested that one reason for the learning difficulties of these complicated concepts is not only the complexity of these concept but the nature of prior knowledge of students where the primitive intuition of continuity is in conflict with the primitive understanding of limit.

Introduction

Mathematics is supposed to have a completely hierarchical structure where all new concepts logically follow from prior ones. The traditional way to teach the concept of continuity starts with the teaching of the concept of limit, because the concept of continuity is based on the concept of limit. Teaching in the traditional way has some advantages but it does not seem to work for the majority of students (Dreyfus 1991). The cognitive process of concept acquisition does not seem to follow the mathematical logic. It is possible that the harmonious whole and logical structure of mathematics appears logical and continuous only for the experts and teachers of mathematics but for the student who is struggling to understand it appears as fragmentary and discontinuous. (Lehtinen, Merenluoto & Kasanen).

The theories of conceptual change (Carey & Spelke 1994, Chi, Slotta, & de Leeuw, 1994, Ferrari & Chi 1998, Vosniadou & Ioannides 1998, diSessa 1993) **stress the relationship** between the prior knowledge and the information. According to these theories it seems obvious that the relationship between prior knowledge and information that ought to be learned is one of the most crucial factors in determining the quality of learning. These theories explain the role of prior knowledge in slightly different ways.

Carey and Spelke (1994) argue that human reasoning is guided by a collection of innate domain-specific systems of knowledge. According to this hypothesis, each system is characterized by a set of core principles that define the entities covered by the domain and support reasoning about those entities. Spelke (1991) has described the continuity as one of the constraints about the behavior of physical objects which infants appear to appreciate from early on. This intuition of continuity is then strongly linked with dynamics of motion. Chi (Chi et al. 1994, Ferrari & Chi 1998) has developed a theory of conceptual change which is based on the philosophical analysis of *ontological categories*. This theory assumes that learning is difficult and may cause misconceptions if a shift from a main category to another is needed. Overcoming these ontological obstacles in the learning process may be very difficult also because we are easily blind by own ontological beliefs (Sfard & Linchevski 1994) According to Vosniadou the naive framework theory of children consist of *ontological and epistemological* presuppositions. These fundamental presumptions constrain the way children interpret their observations to generate a set of interrelated beliefs which are used to explain the phenomena. (Vosniadou & Ioannides 1998.)

DiSessa (1993) gives a framework for describing and correlating characteristics of weakly organized knowledge systems. In dealing with the physical world, students gradually acquire an elaborate *sense of mechanism* – a sense how things work, what sorts of events are necessary, likely, possible, or impossible. (diSessa 1993, 106) This sense of mechanism provides children to asses the possibility of various happenings based on generalizations about what does and does not happen. And it provides the ability to make some predictions and causal descriptions. One description of the sense of mechanism is *causality*. In observing the everyday happenings children observe which events follow others regularly and what is possible and what is not. The intuitive knowledge is composed of phenomenological primitives, which are rather small knowledge structures, typically having only a few parts, that act largely by being **recognized** in a physical system. These primitives may be self-explanatory, something happens "because this is the way things are". The recognition occurs in layers, the way a particular p-prim's transition to an active state is affected by other previously activated elements is called *cuing priority*. High or low cuig priority indicates a stronger or weaker connection between structures that are given the preference. The **drastic revision** in the intuitive knowledge system is in that the function of p-prims changes. In the learning process they start to change from being self-explanatory to much more complex knowledge structure with scientific explanations. According to diSessa (1993) the development of scientific knowledge about the physical world is possible only through reorganized intuitive knowledge.

Earlier the theories of conceptual change have mainly been used to explain the learning problems in the field of science learning (Vosniadou & Ioannides 1998), it seems reasonable to suggest that these theories also explain the difficulties in learning of mathematics, specially the concept of continuity. There are three observations that seem to refer to that direction: **Firstly**, the nature of mathematical conceptions is both operational and structural where operational conception naturally precedes a structural one and where there is a deep ontological gap between them (Sfard & Linchevski 1994). **Secondly**, there was a long development period between the operational use and the structural formalization of these concepts in the history of mathematics (Boyer 1994, Kline 1980). **Thirdly**, the high level of abstraction and complexity in the nature of the concepts of advanced mathematics (Dreyfus 1991, Tall 1991) and the low nature of abstraction of the mathematics in the students everyday life. The concept of continuity is fundamental in the analysis but it also involves understanding the concept of limit, because the definition of continuity is based on the concept of limit. These concepts have a long and difficult history and they are difficult for students today (Lehtinen, Merenluoto & Kasanen 1997, Kaput 1994, Tall & Vinner 1991).

Mathematical experience in practical everyday activities plays an important role in the development of mathematical thinking. From mathematical point of view this is, however, problematic because it systematically reinforces very limited beliefs of what mathematics is about. The children cannot learn, by observing the world, that their initial system of knowledge is false or insufficient (Carey & Spelke 1994). And besides, the language of mathematics tries to describe all the concepts in a very economical way. All the concepts however get their meaning from a large and often highly abstract system of interrelated ideas. The innate principles can foster learning but they can also serve as barriers (Gelman & Brenneman 1994).

Although the theories of conceptual change differ in the way they explain prior knowledge, all of them define two levels of difficulty in the learning process. The easier level of conceptual change means *enrichment* of one's prior knowledge structure. This kind of learning is acquisition of statements and facts. The accumulation of information is a necessary prerequisite for learning, but there are mathematical concepts however which cannot be understood on a basis of prior formal knowledge and on an accumulation of knowledge only but these concepts seem to presuppose the construction of an abstract systemic "environment". Especially in learning these kinds of concepts the more difficult conceptual change is needed, where the student makes a radical reorganization or change in prior knowledge. Sometimes students do not see or understand the reason to change their prior knowledge and logic even though *revision* would be necessary. Then they attempt to synthesize the currently accepted scientific information with aspects of their initial concept. (Vosniadou & Ioannides 1998.) **Our aim** in this report is to explain the role of prior knowledge in the problems the students have in understanding the definition of continuity of function.

Research design

A questionnaire was filled in an usual classroom situation by 640 students (age 17-18 years) of extended mathematics in 24 randomly selected Finnish upper secondary schools in spring 1998. Of this group 272 students answered to same questions half a year later. In this paper we analyze the responses to the two questions pertaining the continuity and limit of a function.

Analysis of responses

In the questionnaire presented to the students they where asked to describe in their own words what is meant with the concepts of continuity and limit of a function. The questions were difficult for the students, so there were 295 students who answered to both of these questions in the first group (group I) and 199 in the second group (group II). In our analysis we use only those students who answered to both of these questions. The framework described by diSessa (1993) seems to describe best the features of prior knowledge we found in the students responses. We classified the answers of the students in six different categories based on the level of abstraction which was seen in the responses (see Cifarelli 1988, Goodson-Espy 1998) and the level of prior everyday experienced based knowledge.

I. The case of continuity

Level 0. These responses indicate that the question was not considered or there were serious misconceptions. For example: *"some number is continuous at a point? (will get clearer)"(207)*[1], *"a function does not cut the x-axis" (190)* or *"a function is monotonous at every point"(453)*.

Level 1. **Primitive** level, where the response was based on everyday experiences and/or naive primitive intuition. The explanations had an causal meaning, a 'sense of mechanism' something continues because it is in motion or does not have an end. *"A function continues towards infinity"(689), "it does not have an end"(245)*. Or where the naive intuition was combined with the scientific explanation: *"a function is continuous if it does not have limits"(339)*

Level 2. The level of **recognition**, where continuity was still cued with motion and everyday experiences but there were indications of recognition of continuity as a function of graph having 'no gaps'. For example: *"a function is continuous if it does not jump"(60), "if the graph does not brake"*(70) or operational everyday experience based answers like *"it is possible to draw without lifting a pen"(766)*.

Level 3. The level of **re-presentation**, where there is some flexibility in recognition and a beginning of transition from the intuition of motion to describing the continuity at a point or connecting the continuity with derivative For example: *"there is no point where the function is not defined"(94), "the function has a derivative"(305).*

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[1] the identification number of the student

Level 4. The level of **structural abstraction**, where the concept of continuity was cued with the abstraction of limit but still indefinite. The abstraction from the level of re-presentation to the structural abstraction needs a radical reorganization of students prior primitives. This drastic difference is also seen in the distribution of answers (table 1). On this level they start to explain the continuity with the concept of limit. For example: "*a function has both value and limit at the point where it is continuous*" *(357).*

Level 5. The level of **structural awareness**, where the students seemed to be aware of the structure of mathematical continuity based on the concept of limit. For example: *"The function is continuous when the limit of a function is the same as it's value at a point"(817).*

II. The case of limit

Level 0. These responses indicate that the question was not considered or there were serious misconceptions. For example: *"somewhere there is a change"(95), "where a graph cuts the x-axis" (205).*

Level 1. **Primitive** level, where the response was based on everyday experiences and/or naive primitive intuition. The explanations had an causal meaning, the mechanism of limit is to stop the motion. For example: *"it ends there"(498), "it is impossible to go over or under you have to turn back"(2167).* Or where the limit is limiting the function from getting higher (or lower) values: *"a biggest or lowest value the function gets*" *(759).*

Level 2. The level of **recognition**, where there were indications of recognition a feature of the concept of limit cued with the dynamics of approaching. For example: *"a value that a function approaches but does not reach"(392),* "*the values of a function approach the limit but do not 'touch' it"(830).* Or they were operational answers where the student cued the derivative to the word limit: *"there is a limit if a function has a derivative"(244).*

Level 3. The level of **re-presentation**, where there is some flexibility in recognition and a beginning of transition from the intuition of the dynamics of approaching to a more definitive explanation. For example: *"A function approaches to a value while x approaches to some value"(911).*

Level 4. The level of **structural abstraction**, where the concept of the limit of a function was cued with the abstraction of approaching from both sides, but still indefinite. For example: *"while approaching from both sides one does or doesn't get the same value, and thus one is able to conclude is the function has a limit" (2250)*

Level 5. The level of **structural awareness**, where there were indications of awareness of the structure of the limit concept. For example: *"One considers the left and right hand limits and if they are the same the function has a limit, the limit is the value of the function at that point."(2136).*

The level of the response	The case of continuity				The case of limit			
	Group I		Group II		Group I		Group II	
	N	%	N	%	N	%	N	%
Misconception	62	21,0	19	9,5	33	11,2	15	7,5
Primitive	32	10,8	11	5,5	129	43,7	88	44,2
Recognition	86	29,2	70	35,3	79	26,8	65	32,8
Re-presentation	102	34,6	79	39,7	40	13,6	18	9,0
Structural abstraction	6	2,0	11	5,5	14	4,7	8	4,0
Structural awareness	7	2,4	9	4,5	0	0	5	2,5
TOTAL	295	100	199	100	295	100	199	100

Table I. The distribution of the responses

The distributions of the answers between the groups differed in the case of continuity in a statistically significant way (χ^2 =26.72, df=10, p=0.003) but not in the case of limit (χ^2= 15,90, df = 10, p = 0.102). This is an indication of the difficulty of the concept of limit but the 'conflicting' nature of the concepts of continuity and limit is clearly seen in the difference of the percentage between the level of re-presentation and structural abstraction. On the level of structural abstraction the concept of limit is connected with the concept of continuity. And the difficulty of the concept of limit is seen also where the distributions of the answers differed in a statistically significant way between the concepts in group I (χ^2=30,946, df = 16, p = 0.014) and in group II (=46,40, df = 25, p = 0.006). An indication of this same phenomena is obvious also in the estimations of certainty of the students. In the case of continuity the means of estimations drop on the transition level, but grow significantly higher on the levels of structural abstraction and awareness. However in the case of the limit the means of estimations of certainty drop in the higher level of answers.

The prior naive intuition of continuity as something that is in motion and the prior intuition of limit as something that stops the motion are in an obvious conflict with each other. The conceptual change from prior knowledge to the mathematical definition of continuity defined by the limit demands a drastic restructure in the prior knowledge. One of the students wrote an answer that describes well the 'shaking' quality of the conceptual change needed: *"after a limit something shaking happens"* (2076).

Conclusion

In the case of continuity about half of group I and 40% of group II based their answers on their prior knowledge of continuity connected with motion, and 35% of group I and 40% of group II were on the transition level to higher level of understanding. A significant progress could be seen in the responses of the students. In the case of limit however almost three quarters of group I and group II based their answers on their prior understanding of limit as a *'limitor'*. Only about one tenth of group I and II were on the transition level. A great majority of those who did not answer to both of these questions seemed to be even more unsure.

The concept of limit is very complicated and difficult. We claim however that the difference is not only due to the complexity of the concept to be learned, but also in the nature of the prior knowledge and in the 'conflict' of everyday intuition linked to those words. The theory of phenomenological primitives (diSessa 1993) seems to describe best the prior knowledge of the students in these concepts. The primitives of motion are cued to the word of continuity and the primitives of stopping as the cause of limit is cued to the word limit in the lower levels of answers. This conflict seems to support systematically the misconceptions of the students.

Moreover the traditional teaching order of these concepts seems also to support the prior knowledge of the students. When the concept of limit is taught in the context of discontinuous functions or functions that are not defined at some points, the prior intuition of limit as a *'limitor'* is supported by the teaching order. The students might benefit from a new kind of approach in the teaching of these concepts. Theoretical approaches developed in the tradition conceptual change could help to develop more explicit methods for supporting adequate conceptual revision in students.

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